

Level - 1

- 1.(A) The determinant f is symmetric in x, y, z and vanishes when x is replaced by y or z and also when y is replaced by z so that $f = (x - y)(y - z)(z - x) F(x, y, z)$. $F(x, y, z)$ being a symmetric function in x, y, z .

The main diagonal gives xy^3z and in the product, we have the term $(x - y)(y - z)(z - x)$. It follows that:

$$F(x, y, z) = xy + yz + zx$$

$$f = (x - y)(y - z)(z - x)(xy + yz + zx) \quad \text{and} \quad g = (x - y)(y - z)(z - x)$$

$$\Rightarrow \frac{f}{g} = xy + yz + zx$$

$$2.(A) \quad g(x) = \begin{vmatrix} a^{-x} & e^{\log_e a^x} & x^2 \\ a^{-3x} & e^{\log_e a^{3x}} & x^4 \\ a^{-5x} & e^{\log_e a^{5x}} & 1 \end{vmatrix} = \begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix} \quad \left(\because e^{\log a^x} = a^x \right)$$

$$g(-x) = \begin{vmatrix} a^x & a^{-x} & x^2 \\ a^{3x} & a^{-3x} & x^4 \\ a^{5x} & a^{-5x} & 1 \end{vmatrix} = - \begin{vmatrix} a^{-x} & a^x & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix} \quad \left(\begin{array}{l} \text{Interchanging 1}^{\text{st}} \\ \text{and 2}^{\text{nd}} \text{ columns} \end{array} \right) = -g(x) \Rightarrow g(x) + g(-x) = 0$$

$$3.(C) \quad \begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 1+3z \end{vmatrix} = xyz \begin{vmatrix} 1+\frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

Use, $R_1 \rightarrow R_1 + R_2 + R_3$

$$= xyz \left(3 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \begin{vmatrix} 1 & 1 & 1 \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

Use, $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$= xyz \left(3 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \begin{vmatrix} 1 & 0 & 0 \\ 1+\frac{1}{y} & 1 & -1 \\ 1+\frac{1}{z} & 0 & 2 \end{vmatrix} = 2xyz \left(3 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = 0$$

$$\Rightarrow x^{-1} + y^{-1} + z^{-1} = -3$$

$$4.(A) \quad A^2 = AA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix}$$

Again

$$\begin{aligned} A^3 &= A^2A = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} = \begin{bmatrix} p & q & r \\ pr & p+qr & q+r^2 \\ pq+r^2p & pr+q^2+qr^2 & p+2qr+r^3 \end{bmatrix} \\ &= \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & qr \end{bmatrix} + \begin{bmatrix} 0 & 0 & r \\ pr & qr & r^2 \\ pr^2 & pr+qr^2 & qr+r^3 \end{bmatrix} \\ &= p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + q \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} + r \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix} \\ &= pI + qA + rA^2 \quad \Rightarrow \quad A^3 - rA^2 - qA = pI \end{aligned}$$

$$5.(AB) \quad 2x - y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix} \quad \dots (i) \quad \text{and} \quad x + 2y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} \quad \dots (ii)$$

Multiply (i) by 2 and add to (ii) to get :

$$5x = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Multiplying (ii) by 2 and subtract (i) from it to get:

$$5y = \begin{bmatrix} 8 & 2 & 10 \\ -2 & 8 & -8 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 10 \\ -5 & 5 & -10 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \quad x + y = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix} \quad \text{and} \quad x - y = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$6.(C) \quad f(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x) \cdot f(y)$$

$$7.(B) \quad \cos^2 \theta \cos^2 \phi + \cos \theta \sin \theta \cos \phi \sin \phi = 0$$

$$\Rightarrow \quad \cos \theta \cos \phi [\cos \theta \cos \phi + \sin \theta \sin \phi] = 0 \Rightarrow \cos \theta \cos \phi \cos(\theta - \phi) = 0 \Rightarrow \theta - \phi = \frac{\pi}{2}$$

$$8.(A) \quad A + B = \begin{bmatrix} -\omega^2 & -1 & -\omega \\ -1 & -\omega & -\omega^2 \\ -1 & -\omega & -\omega^2 \end{bmatrix}$$

$$(A + B)C = \begin{bmatrix} -\omega^2 & -1 & -\omega \\ -1 & -\omega & -\omega^2 \\ -1 & -\omega & -\omega^2 \end{bmatrix} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$9.(A) \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$$10.(A) \begin{bmatrix} x+y & y \\ 2x & x-y \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+y \\ 3x+y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} 2x+y=3 \\ 3x+y=2 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=5 \end{cases} \Rightarrow xy=-5$$

$$11.(B) \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0 \Rightarrow bc + ca + ab = 0$$

$$(1+a)[(1+b)(1+c)-1] - [1+c-1] + [1-1-b] = \lambda$$

$$\Rightarrow (1+a)[b+c+bc] - c - b = \lambda$$

$$\Rightarrow b+c+bc+ab+ac+abc-c-b = \lambda \Rightarrow bc+ab+ca+abc = \lambda \Rightarrow \lambda = abc$$

$$12.(C) \text{ we have, } ax^4 + bx^3 + cx^2 + 50x + d = \begin{vmatrix} x^3 - 14x^2 & -x & 3x + \lambda \\ 4x + 1 & 3x & x - 4 \\ -3 & 4 & 0 \end{vmatrix}$$

On differentiating with respect to x.

$$4ax^3 + 3bx^2 + 2cx + 50 = \begin{vmatrix} 3x^2 - 28x & -1 & 3 \\ 4x + 1 & 3x & x - 4 \\ -3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} x^3 - 14x^2 & -x & 3x + \lambda \\ 4 & 3 & 1 \\ -3 & 4 & 0 \end{vmatrix}$$

$$50 = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & \lambda \\ 4 & 3 & 1 \\ -3 & 4 & 0 \end{vmatrix} \Rightarrow 50 = 25\lambda \Rightarrow \lambda = 2$$

$$13.(B) \begin{vmatrix} 1 + \sin^2 \theta & \cos^2 \theta & 4 \sin 4\theta \\ \sin^2 \theta & 1 + \cos^2 \theta & 4 \sin 4\theta \\ \sin^2 \theta & \cos^2 \theta & 1 + 4 \sin 4\theta \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \sin^2 \theta & \cos^2 \theta & 1 + 4 \sin 4\theta \end{vmatrix} \begin{pmatrix} \because R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 - R_3 \end{pmatrix}$$

$$= 1 + 4 \sin 4\theta + \cos^2 \theta + \sin^2 \theta = 2 + 4 \sin 4\theta = 0 \Rightarrow \sin 4\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$$

$$14.(A) f(x) = x^3 - 9x^2 + 27x - 23 = 0$$

$$f'(x) = 3x^2 - 18x + 27 \Rightarrow D = 0$$

$$15.(B) AB = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \sin \phi \cos \phi \\ \sin \phi \cos \phi & \sin^2 \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos \phi \cos(\theta - \phi) & \cos \theta \sin \phi \cos(\theta - \phi) \\ \cos \theta \sin \phi \cos(\theta - \phi) & \sin \theta \sin \phi \cos(\theta - \phi) \end{bmatrix} = \cos(\theta - \phi) \begin{bmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi \\ \cos \theta \sin \phi & \sin \theta \sin \phi \end{bmatrix} = O, \left(\because \theta - \phi = \frac{\pi}{2} \right).$$

$$16.(B) \Delta = (2+i) \begin{vmatrix} 1 & 1 & i \\ 1 & 1+2i & 1+i \\ 1 & 2 & 1-i \end{vmatrix} = (2+i) \begin{vmatrix} 0 & -2i & -1 \\ 0 & -1+2i & 2i \\ 1 & 2 & 1-i \end{vmatrix} \text{ by } \begin{matrix} R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 - R_3 \end{matrix}$$

$$= (2+i) \{-4i^2 + (-1+2i)\} = (2+i)(4-1+2i) = (2+i)(3+2i) = 4+7i.$$

$$\begin{aligned}
 17.(D) \quad \Delta &= \begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = \begin{vmatrix} x+1+\omega+\omega^2 & \omega & \omega^2 \\ x+1+\omega+\omega^2 & x+\omega^2 & 1 \\ x+1+\omega+\omega^2 & 1 & x+\omega \end{vmatrix}, (C_1 \rightarrow C_1 + C_2 + C_3) \\
 &= x \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x+\omega^2 & 1 \\ 1 & 1 & x+\omega \end{vmatrix} \quad (\because 1+\omega+\omega^2=0) \\
 &= x[1\{(x+\omega^2)(x+\omega)-1\} + \omega\{1-(x+\omega)\} + \omega^2\{1-(x+\omega^2)\}] \\
 &= x(x^2 + \omega x + \omega^2 x + \omega^3 - 1 + \omega - \omega x - \omega^2 + \omega^2 - \omega^2 x - \omega^4) = x^3, \quad (\because \omega^3 = 1).
 \end{aligned}$$

$$\begin{aligned}
 18.(B) \quad \begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} &= (x+y+z) \begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix} \\
 \text{By } R_1 &\rightarrow R_1 + R_2 + R_3 \\
 &= (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & z & x \\ x & y & z \end{vmatrix}; \text{ by } C_1 \rightarrow C_1 - C_2 = (x+y+z) \left\{ (z^2 - xy) - (xz - x^2) + (xy - xz) \right\} \\
 &= (x+y+z)(x-z)^2 \Rightarrow k=1.
 \end{aligned}$$

$$\begin{aligned}
 19.(A) \quad \begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow (x+9) \begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0, \text{ By } R_1 \rightarrow R_1 + R_2 + R_3 \\
 \Rightarrow (x+9)\{(x^2-12)-(2x-14)+(12-7x)\} = 0 \Rightarrow (x+9)(x^2-9x+14) = 0 \Rightarrow (x+9)(x-2)(x-7) = 0 \\
 \text{Hence the other two roots are } x=2, 7
 \end{aligned}$$

20.(D) Fundamental concept.

$$\begin{aligned}
 21.(B) \quad \Delta &= \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} \\
 \text{By } R_1 &\rightarrow R_1 + R_2 + R_3 \\
 \Delta &= (a+b+c) \begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} \\
 \text{On expanding, } &-(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = -(a^3+b^3+c^3-3abc) = 3abc - a^3 - b^3 - c^3.
 \end{aligned}$$

22.(B) Since $\Delta = \omega^2 - 2\omega^2 = -\omega^2$. Therefore $\Delta^2 = \omega^4 = \omega$.

$$23.(B) \quad \Delta_2 \Delta_1 = \begin{vmatrix} 1 & 0 \\ c & d \end{vmatrix} \begin{vmatrix} 1 & 0 \\ a & b \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ c+ad & bd \end{vmatrix} = bd.$$

$$\begin{aligned}
 24.(B) \quad A^2 = AA &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 2 \\ 3 & 4 & 1 \end{bmatrix} \Rightarrow A^3 = A^2 A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 6 & 2 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 3 \\ 15 & 19 & 6 \\ 9 & 12 & 4 \end{bmatrix} \\
 \text{Here, } A^3 - 3A^2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \Rightarrow A^3 - 3A^2 - I = 0
 \end{aligned}$$

25.(B) Given, $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \quad B^2 = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

Also, $(A+B)^2 = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix} \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$

$$(A+B)^2 = \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(1+a)-2(2+b) & 4 \end{bmatrix}$$

$$\therefore (A+B)^2 = A^2 + B^2 \Rightarrow \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

By equating, $a-1=0 \Rightarrow a=1$ and $b=4$.

26.(B) $A^2 = A.A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$ and $5A = \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix}$

$$\therefore A^2 - 5A = 14 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 14I.$$

27.(D) Given, Matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We know that $A^2 = A.A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Therefore $A^{16} = (A^2)^8 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^8 = \begin{bmatrix} (-1)^8 & 0 \\ 0 & (-1)^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

28.(B) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^3 = 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A^{100} = 2^{99} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

29.(C) Since $A^2 = A.A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}$

And $A^3 = \begin{bmatrix} a^3 & 0 & 0 \\ 0 & b^3 & 0 \\ 0 & 0 & c^3 \end{bmatrix}, \dots$

$$\Rightarrow A^n = A^{n-1} \cdot A = \begin{bmatrix} a^{n-1} & 0 & 0 \\ 0 & b^{n-1} & 0 \\ 0 & 0 & c^{n-1} \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}.$$

30.(A) The given matrix $A = \begin{bmatrix} 2 & \lambda & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is non singular, if $|A| \neq 0$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & \lambda & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & \lambda+3 & 0 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{vmatrix}, [R_1 \rightarrow R_2 + R_1] \\ &= \begin{vmatrix} 1 & \lambda+3 & 0 \\ 0 & 1 & 1 \\ 0 & -\lambda-5 & -3 \end{vmatrix} \begin{matrix} [R_2 \rightarrow R_2 + R_3] \\ [R_3 \rightarrow R_3 - R_1] \end{matrix} \\ &= 1(-3 + \lambda + 5) \neq 0 \Rightarrow \lambda + 2 \neq 0 \Rightarrow \lambda \neq -2. \end{aligned}$$

31.(A) $\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix}$

$$\begin{aligned} \text{Use, } R_2 \rightarrow R_2 - (R_1 + R_3) & \text{ gives } \begin{vmatrix} x^2 + x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} = 12(2x-1) = Ax + B \\ \Rightarrow A = 24, B = -12 & \Rightarrow A + 2B = 0 \end{aligned}$$

32.(D) We have

$$\begin{aligned} \phi'(x) &= \begin{vmatrix} f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ f^{iv}(x) & g^{iv}(x) & h^{iv}(x) \end{vmatrix} \\ &= 0 + 0 + \begin{vmatrix} f'(x) & g'(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad [\because f, g, h \text{ are polynomials of degree 3, } f^{iv}(x) = g^{iv}(x) = h^{iv}(x) = 0] \\ \Rightarrow \phi(x) & \text{ must be a constant.} \end{aligned}$$

33.(D) We have, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Similarly, $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$. Now, go option by option

Clearly A and C are ruled out.

(B) $\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \neq \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} + \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

(D) Now, $nA - (n-1)I = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

34.(C) Now, $P^3 = P(I - P)$

$$= PI - P^2 = PI - (I - P) = P - I + P = 2P - I$$

$$P^4 = P \cdot P^3 \Rightarrow P^4 = P(2P - I) \Rightarrow P^4 = 2P^2 - P$$

$$\Rightarrow P^4 = 2I - 2P - P \Rightarrow P^4 = 2I - 3P \Rightarrow P^5 = P(2I - 3P)$$

$$\Rightarrow P^5 = 2P - 3(I - P) \Rightarrow P^5 = 5P - 3I \Rightarrow P^6 = P(5P - 3I)$$

$$\Rightarrow P^6 = 5P^2 - 3P \Rightarrow P^6 = 5(I - P) - 3P \Rightarrow P^6 = 5I - 8P, \text{ So, } n = 6.$$

$$35.(A) \quad 3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix} \Rightarrow A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{x}{3} & \frac{2}{3} & \frac{y}{3} \end{bmatrix} \Rightarrow AA^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{x}{3} & \frac{2}{3} & \frac{y}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{x}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{y}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{x+2y+4}{9} \\ 0 & 1 & \frac{2x-2y+2}{9} \\ \frac{x+2y+4}{9} & \frac{2x-2y+2}{9} & \frac{x^2+4+y^2}{9} \end{bmatrix}$$

$$\Rightarrow x + 2y + 4 = 0 \quad \dots (i) \quad 2x - 2y + 2 = 0 \quad \dots (ii)$$

$$\text{Solve (i) and (ii) to get : } x = -2 \quad \text{and} \quad y = -1 \Rightarrow x + y = -3$$

$$36.(ABD) \quad A^2 - 4A - 5I_3 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \Rightarrow A^2 - 4A - 5I_3 = 0$$

$$\Rightarrow A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I_3 = 0 \Rightarrow (A^{-1}A)A - 4I_3 - 5A^{-1} = 0$$

$$\Rightarrow IA - 4I_3 - 5A^{-1} = 0 \quad \therefore A^{-1} = \frac{1}{5}(A - 4I_3)$$

$$\text{Also, } |A^2| = \begin{vmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{vmatrix} = 25 \neq 0 \quad \therefore A^2 \text{ is invertible}$$

$$\text{and } A^3 = A \cdot A^2 = A \cdot (4A + 5I_3) = 4A^2 + 5A$$

$$= \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} + \begin{bmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$$

$$\Rightarrow |A^3| \neq 0 \Rightarrow A^3 \text{ is invertible.}$$

$$37.(A) \quad P^T P = PP^T = I \Rightarrow Q^2 = PAP^T PAP^T = PA^2 P^T \Rightarrow Q^n = PA^n P^T$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \Rightarrow A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \Rightarrow P^T Q^{2005} P = P^T PA^{2005} P^T P = A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

$$38.(BC) \quad z = e^{iA} \cdot e^{iB} \cdot e^{iC} \begin{vmatrix} e^{iA} & e^{-i(C+A)} & e^{-i(B+A)} \\ e^{-i(C+B)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix} = -1 \cdot \begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

Because $e^{i(A+B+C)} = e^{i\pi} = \cos \pi + i \sin \pi = -1$, etc.

$$= -e^{i(A+B+C)} \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4$$

$$39.(A) \quad \begin{vmatrix} 1 & 1 & 1 \\ (2^x + 2^{-x})^2 & (3^x + 3^{-x})^2 & (5^x + 5^{-x})^2 \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix}$$

Operate $R_2 \rightarrow R_2 - R_3 - 4R_1$ to get:

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix} = 0$$

$$40.(D) \quad \text{Since, } A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\therefore B = \text{adj } A = \begin{bmatrix} 3 & 1 & 1 \\ -6 & -2 & 3 \\ -4 & -3 & 2 \end{bmatrix}$$

$$\Rightarrow \text{adj } B = \begin{bmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{bmatrix} \Rightarrow |\text{adj } B| = \begin{vmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{vmatrix} = 625$$

Given that $C = 5A$

$$\Rightarrow |C| = 5^3 |A| = 5^3 \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{vmatrix} = 625 \Rightarrow \frac{|\text{adj } B|}{|C|} = 1$$

$$41.(A) \quad f(x) = \begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix} \quad (\text{differentiating column by column})$$

$$f'(x) = \begin{vmatrix} 1 & 2x & 0 \\ 1 & 4x & 0 \\ 1 & 6x & 0 \end{vmatrix} + \begin{vmatrix} x+a & 2x & 1 \\ x+b & 2x^2-1 & 0 \\ x+c & 3x^2-2 & 0 \end{vmatrix} + \begin{vmatrix} x+a & x^2+1 & 0 \\ x+b & 2x^2-1 & 0 \\ x+c & 3x^2-2 & 0 \end{vmatrix} = 0 + 2x \begin{vmatrix} x+a & 1 & 1 \\ b-a & 1 & 0 \\ c-b & 1 & 0 \end{vmatrix} + 0 \begin{pmatrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{pmatrix}$$

$$= 2x(2b-a-c) = 0 \quad (\because a, b, c \text{ are in A.P.})$$

42.(CD) Given: $\begin{vmatrix} x & 2 & 2 \\ 3 & x & 2 \\ 3 & 3 & x \end{vmatrix} + \begin{vmatrix} 1-x & 2 & 4 \\ 2 & 4-x & 8 \\ 4 & 8 & 16-x \end{vmatrix} > 33$

$$\Rightarrow \Delta_1 + \Delta_2 > 33 \quad \dots (i)$$

$$\Delta_1 = \begin{vmatrix} x & 2 & 2 \\ 3 & x & 2 \\ 3 & 3 & x \end{vmatrix} = x(x^2 - 6) - 2(3x - 6) + 2(9 - 3x) = x^3 - 18x + 30$$

$$\Delta_2 = \begin{vmatrix} 1-x & 2 & 4 \\ 2 & 4-x & 8 \\ 4 & 8 & 16-x \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 4-x & 8 \\ 4 & 8 & 16-x \end{vmatrix} - \begin{vmatrix} x & 2 & 4 \\ 0 & 4-x & 8 \\ 0 & 8 & 16-x \end{vmatrix}$$

Operate $C_2 \rightarrow C_2 - 2C_1; C_3 \rightarrow C_3 - 4C_1$ in first determinant

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2 & -x & 0 \\ 4 & 0 & -x \end{vmatrix} - x \begin{vmatrix} 1 & 2 & 4 \\ 0 & 4-x & 8 \\ 0 & 8 & 16-x \end{vmatrix} = 21x^2 - x^3$$

$\therefore$ From equation (i), we get

$$x^3 - 18x + 30 + 21x^2 - x^3 > 33$$

$$\Rightarrow 21x^2 - 18x - 3 > 0 \Rightarrow 3(7x+1)(x-1) > 0 \Rightarrow x \in \left(-\infty, -\frac{1}{7}\right) \cup (1, \infty)$$

43.(AC) $f(n) = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} n & 1 & 1 \\ n! & nn! & (n+1)(n+1)! \\ 1 & 0 & 0 \end{vmatrix} \quad \left(\text{using : } \begin{matrix} C_3 \rightarrow C_3 - C_2 \\ C_2 \rightarrow C_2 - C_1 \end{matrix} \right)$$

$$= n!(n^2 + n + 1)$$

Thus, $f(n)$ is divisible by $n!$ and $n^2 + n + 1$

44.(D) $\Delta = 0$

$$\Rightarrow f(x) f\left(\frac{1}{x}\right) - f(x) - f\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \Rightarrow \left(a + bx^n\right)\left(a + \frac{b}{x^n}\right) = \left(a + bx^n\right) + \left(a + \frac{b}{x^n}\right)$$

Comparing coeffs. Of x^n on both sides,

$$ab = b \Rightarrow a = 1$$

Equating constant terms on both sides.

$$a^2 + b^2 = 2a$$

$$\Rightarrow 1 + b^2 = 2 \Rightarrow b^2 = 1 \Rightarrow b = \pm 1$$

$$\therefore f(x) = 1 \pm x^n$$

$$f(2) = 17 = 1 + (2)^4 \Rightarrow n = 4$$

$$\therefore f(5) = 1 + (5)^4 = 626$$

45.(B) We have $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 5x + 6 & x + 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 + 5x + 6 + x^2 + 4x \end{bmatrix} = 0$$

$$\Rightarrow x^2 + 9x + 7 = 0 \Rightarrow x = \frac{-9 \pm \sqrt{53}}{2}$$

46.(C) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 0 - 1[1 + 3] - 1[1 + 3] = -4 - 4 = -8 \neq 0 \quad \therefore \text{rank} = 3$

47.(C) $R_3 \rightarrow R_3 - 3R_1 \quad R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\therefore \text{rank} = 3$$

48.(C) $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 1 \\ 3 & 9 & 12 & 9 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \quad R_3 \rightarrow R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

49.(A) $\begin{vmatrix} 1 & 2 & 3 \\ \lambda & 2 & 4 \\ 2 & -3 & 1 \end{vmatrix} \neq 0$

$$\Rightarrow 14 - 2(\lambda - 8) + 3(-3\lambda - 4) \neq 0 \Rightarrow 18 - 11\lambda \neq 0 \Rightarrow \lambda \neq \frac{18}{11}$$

50.(A) $R_3 \rightarrow R_3 - 3R_1$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{rank} = 1$$

51.(B) Since $A^{-1} = \frac{\text{adj } A}{|A|}$ and if λ is eigen value of A , then λ^{-1} is eigen value of A^{-1} .

Thus for $\text{adj}(A)X = (A^{-1}X)|A| = \lambda^{-1}I$

Thus, eigen value corresponding to $\lambda = 3$ is $4/3$ and corresponding to $\lambda = -2$ is $4/-2 = -2$.

52.(A) $1 + P + P^2 + \dots + P^n = O$

Pre-multiplying both sides by P^{-1}

$$\Rightarrow P^{-1} + I + IP + \dots + P^{n-1}I = O.P^{-1} \Rightarrow P^{-1} + I(1 + P + P^2 + \dots + P^{n-1}) = O$$

$$\Rightarrow P^{-1} = -(1 + P + P^2 + \dots + P^{n-1})I \Rightarrow P^{-1} = -(-P^n) = P^n$$

53.(A) $P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{3 \times 3} \begin{bmatrix} -1 & -2 \\ -2 & 0 \\ 0 & -4 \end{bmatrix}_{3 \times 2} \begin{bmatrix} -4 & -5 & -6 \\ 0 & 0 & 1 \end{bmatrix}_{2 \times 3}$

$$P = \begin{bmatrix} -3 & -14 \\ -8 & -20 \\ -11 & -26 \end{bmatrix}_{3 \times 2} \begin{bmatrix} -4 & -5 & -6 \\ 0 & 0 & 1 \end{bmatrix}_{2 \times 3}$$

$$P = \begin{bmatrix} 12 & 15 & 4 \\ 32 & 40 & 28 \\ 44 & 55 & 40 \end{bmatrix}_{3 \times 3} \Rightarrow P_{22} = 40$$

54.(A) $\Delta = 1(\omega^{3n} - 1) + \omega^n(\omega^{2n} - \omega^{2n}) + \omega^{2n}(\omega^n - \omega^{4n})$

$$\Delta = [(\omega^3)^n - 1] + 0 + \omega^{2n}[\omega^n - (\omega^3)^n \cdot \omega^n]$$

$$\Delta = 1 - 1 + 0 + \omega^{2n}[\omega^n - \omega^n] = 0$$

55.(A) Given that, $I + P + P^2 + \dots + P^n = 0$

Pre - multiplying both side by P^{-1} , we get:

$$P^{-1} + P^{-1}P + (P^{-1}P)P + \dots + (P^{-1}P)P^{n-1} = 0$$

$$\Rightarrow P^{-1} + I + IP + \dots + IP^{n-1} = 0 \Rightarrow P^{-1} = 0 - I(1 + P + P^2 \dots P^{n-1}) \Rightarrow P^{-1} = -I(-P^n) \Rightarrow P^{-1} = P^n$$

56.(C) $D = \text{diagonal } [d_1, d_2, d_3, \dots, d_n]$

$$= \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

As $DD^{-1} = I_n$

$$\Rightarrow D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{d_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{d_n} \end{bmatrix}$$

57.(C) $A \cdot A^T = I_n \Rightarrow A - I_n = A - AA^T = A(I_n - A^T)$

$$\Rightarrow |A - I_n| = |A(I_n - A^T)| = |A| |I_n - A^T| = |A| |I_n - A|$$

58.(C) If A is a square matrix, then $A - A^T$ is a skew-symmetric matrix. Then $|A - A^T|$ is '0' or a perfect square as A is of odd order or even order.

59.(A) in ΔABC , given $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$

$$\Rightarrow 1(c^2 - ab) - a(c - a) + b(b - c) = 0 \Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$$

$$\Rightarrow \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$$

Here, sum of squares of three members can be zero, if only, if $a = b = c$

$$\Rightarrow \Delta ABC \text{ is equilateral} \Rightarrow \angle A = \angle B = \angle C = 60^\circ$$

$$\therefore \sin^2 A + \sin^2 B + \sin^2 C = 3 \sin^2 60^\circ = \frac{9}{4}$$

60.(D) By the law of sines $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$$\Rightarrow a = k \sin A, b = k \sin B, c = k \sin C$$

$$\begin{aligned} \text{Now } \Delta &= \begin{vmatrix} a^2 & ab/k & ac/k \\ ab/k & 1 & \cos(B-C) \\ ac/k & \cos(B-C) & 1 \end{vmatrix} = a^2 \begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos(B-C) \\ \sin C & \cos(B-C) & 1 \end{vmatrix} \\ &= a^2 \begin{vmatrix} 1 & \sin(A+C) & \sin(A+B) \\ \sin(A+C) & 1 & \cos(B-C) \\ \sin(A+B) & \cos(B-C) & 1 \end{vmatrix} = a^2 \begin{vmatrix} \sin A & \cos A & 0 \\ \cos C & \sin C & 0 \\ \cos B & \sin B & 0 \end{vmatrix} \begin{vmatrix} \sin A & \cos A & 0 \\ \cos C & \sin C & 0 \\ \cos B & \sin B & 0 \end{vmatrix} = 0 \end{aligned}$$

61.(A)

62.(C)

63.(B)

64.(C) For $n = 2$

$$\Rightarrow (A^{-1}BA)(A^{-1}BA) = A^{-1}B^2A$$

$$(A^{-1}BA)^3 = (A^{-1}BA)^2 (A^{-1}BA) = (A^{-1}B^2A)(A^{-1}BA) = A^{-1}B^3A \text{ and so on...}$$

$$\text{Thus } (A^{-1}BA)^n = A^{-1}B^nA$$

65. (C) Let $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$. Now $A^T = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$

$$\text{As } A \text{ is orthogonal} \Rightarrow AA^T = I$$

$$\Rightarrow \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$4b^2 + c^2 = 1 \quad \dots \text{(i)}, \quad 2b^2 - c^2 = 0 \dots \text{(ii)} \quad \text{and} \quad a^2 + b^2 + c^2 = 1 \dots \text{(iii)}$$

On solving (i), (ii) and (iii), we get $a = \pm \frac{1}{\sqrt{2}}; b = \pm \frac{1}{\sqrt{6}}; c = \pm \frac{1}{\sqrt{3}}$

66. (B) Solve $2x + y = 5$ and $x + 3y = 5$ to get $x = 2$ and $y = 1$. As $(x, y) \equiv (2, 1)$ satisfies the equation $x - 2y = 0$, we have unique solution for given system of equations.

67-70

67.(D) 68.(D) 69.(ABC) 70.(ABC)

Cofactor matrix of A is
$$\begin{bmatrix} bc - bd & d - c & 0 \\ -c & ac & 1 - ab \\ d & -ad & ab - 1 \end{bmatrix}$$

$$(\text{Adj } A) = \begin{bmatrix} bc - bd & -c & d \\ d - c & ac & -ad \\ 0 & 1 - ab & ab - 1 \end{bmatrix}$$

$$(\text{Adj } A)U = \begin{bmatrix} bc - bd & -c & d \\ d - c & ac & -ad \\ 0 & 1 - ab & ab - 1 \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} bcf - bdf - cg + dh \\ df - fc + acg - agd \\ g - abg + abh - h \end{bmatrix}$$

Cofactor matrix of B is
$$\begin{bmatrix} hd - gc & fc & -fd \\ g - h & ah - f & f - ag \\ c - d & -ac & ad \end{bmatrix}$$

$$(\text{Adj } B)V = \begin{bmatrix} hd - gc & g - h & c - d \\ fc & ah - f & -ac \\ -fd & f - ag & ad \end{bmatrix} \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a^2hd - a^2gc \\ a^2fc \\ -a^2fd \end{bmatrix}$$

From 67

$$= \begin{bmatrix} 0 \\ a^2fc \\ -a^2fd \end{bmatrix}$$

Also $|A| = a(bc - bd) - (c - d) = abc - abd - c + d$

71.(A) $\lambda = \frac{t^2 - 3t + 4}{t^2 + 3t + 4}$

$$\Rightarrow (\lambda - 1)t^2 + 3(\lambda + 1)t + 4(\lambda - 1) = 0$$

Since, t is real

$$\Rightarrow 9(\lambda + 1)^2 - 16(\lambda - 1)^2 \geq 0$$

$$\Rightarrow (3\lambda + 3 - 4\lambda + 4)(3\lambda + 3 + 4\lambda - 4) \geq 0 \Rightarrow (7 - \lambda)(7\lambda - 1) \geq 0 \Rightarrow \frac{1}{7} \leq \lambda \leq 7$$

Now, $\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix}$

[Determinant of coefficients of equations]

$$= 7(\lambda + 5) \neq 0 \quad \left[\because \frac{1}{7} \leq \lambda \leq 7 \right]$$

Hence the given system of equations has a unique solution.

$$72.(A) \text{ Determinant of coefficients} = \begin{vmatrix} \lambda & -1 & \cos \theta \\ 3 & 1 & 2 \\ \cos \theta & 1 & 2 \end{vmatrix} = \cos \theta - \cos^2 \theta + 6$$

And this is positive for all θ since $|\cos \theta| \leq 1$.

The only solution is therefore the trivial solution.

73.(BC) Using $R_3 \rightarrow R_3 - R_2, R_2 \rightarrow R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} x+a & x+b & x+a-c \\ k & k & 2k-1 \\ k & k & 1+2k \end{vmatrix},$$

$$\text{Where } k = \text{common difference} \quad \therefore \Delta = \begin{vmatrix} x+a & k & -c \\ k & 0 & k-1 \\ k & 0 & k+1 \end{vmatrix} \left(\begin{array}{l} \because C_2 \rightarrow C_2 - C_1 \\ \text{and } C_3 \rightarrow C_3 - C_1 \end{array} \right)$$

$$= -k \{k(k+1) - k(k-1)\} = -2k^2$$

$$\therefore \int_0^2 \Delta(x) dx = \int_0^2 -2k^2 dx = -4k^2 = -16 \Rightarrow k = 2, -2$$

74.(AB) The number of third-order determinants = the number of arrangements of nine different numbers in nine places = 9!

Corresponding to each determinant made, there is a determinant obtained by interchanging two consecutive rows or columns). So, the sum of this pair will be 0.

$\therefore$ the sum of all the determinants.

$$= 0 + 0 + 0 + \dots \text{ to } \frac{9!}{2} \text{ times} = 0$$

$$75.(D) \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0 \left(\begin{array}{l} \because R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 - R_3 \end{array} \right)$$

[$\because$ system has non-zero solution also, $D_1 = D_2 = D_3 = 0$ So, D must be equal to zero]

$$\begin{vmatrix} p-q & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0 \Rightarrow (p-a)(q-b)(r-c) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a}{p-a} & \frac{b}{q-b} & \frac{r}{r-c} \end{vmatrix} = 0$$

$$\text{As } p \neq a, q \neq b, r \neq c, \text{ we have } \Rightarrow \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a}{p-a} + \frac{b}{q-b} + \frac{r}{r-c} & \frac{b}{q-b} & \frac{r}{r-c} \end{vmatrix} = 0$$

($\because C_1 \rightarrow C_1 + C_2 + C_3$)

$$\Rightarrow \frac{a}{p-a} + \frac{b}{q-b} + \frac{r}{r-c} = 0 \Rightarrow \left(1 + \frac{a}{p-a}\right) + \left(\frac{b}{q-b} + 1\right) + \left(\frac{r}{r-c}\right) = 1 \Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{c}{r-c} = 1$$

$$76.(B) \quad \Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix} = x^3 - 3abx \Rightarrow \frac{d}{dx} \Delta_1 = 3(x^2 - ab)$$

$$\text{and } \Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix} = x^2 - ab \Rightarrow \frac{d}{dx}(\Delta_1) = 3(x^2 - ab) = 3\Delta_2.$$

77.(A) Ratio of cofactor to its minor of the element -3 , which is in the 3rd row and 2nd column $= (-1)^{3+2} = -1$.

$$78.(D) \quad \Delta^c = \Delta^{n-1} = \Delta^{3-1} = \Delta^2 = (11)^2 = 121.$$

But we have to find the value of the square of the determinant, so required value is $(121)^2 = 14641$.

$$79.(B) \quad \text{From Cramer's rule, } x = \frac{D_x}{D} = \frac{\begin{vmatrix} -d_1 & b_1 & c_1 \\ -d_2 & b_2 & c_2 \\ -d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \Rightarrow x = \frac{-\Delta(bcd)}{\Delta(abc)}.$$

$$80.(C) \quad \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix} = 10! \begin{vmatrix} 1 & 11 & 11 \times 12 \\ 1 & 12 & 12 \times 13 \\ 1 & 13 & 13 \times 14 \end{vmatrix}$$

$$\text{Applying } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1 = 10! \begin{vmatrix} 1 & 11 & 11 \times 12 \\ 0 & 1 & 24 \\ 0 & 2 & 50 \end{vmatrix} = 2(10! \begin{vmatrix} 1 & 11 & 11 \times 12 \\ 0 & 1 & 24 \\ 0 & 2 & 50 \end{vmatrix}).$$

$$81.(B) \quad \text{The cofactor of element 4, in the 2nd row and 3rd column is } = (-1)^{2+3} \begin{vmatrix} 1 & 3 & 1 \\ 8 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -\{1(-2) - 3(8-0) + 1.16\} = 10.$$

$$82.(B) \quad \text{We know that } \Delta \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} \Sigma a_1 A_1 & 0 & 0 \\ 0 & \Sigma a_2 A_2 & 0 \\ 0 & 0 & \Sigma a_3 A_3 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

$$\Rightarrow \Delta' = \Delta^2.$$

83.(D) Fundamental concept

$$84.(A) \quad B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1 c_3 - c_1 a_3 \quad ; \quad C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -(a_1 b_3 - a_3 b_1)$$

$$B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = -(a_1 c_2 - a_2 c_1) \quad ; \quad C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 c_3 - a_3 c_1 & -(a_1 b_3 - a_3 b_1) \\ -(a_1 c_2 - a_2 c_1) & a_1 b_2 - a_2 b_1 \end{vmatrix} = \begin{vmatrix} a_1 c_3 & -a_1 b_3 \\ -a_1 c_2 & a_1 b_2 \end{vmatrix} + \begin{vmatrix} a_1 c_3 & a_3 b_1 \\ -a_1 c_2 & -a_2 b_1 \end{vmatrix} + \begin{vmatrix} -a_3 c_1 & -a_1 b_3 \\ a_2 c_1 & a_1 b_2 \end{vmatrix} + \begin{vmatrix} -a_3 c_1 & a_3 b_1 \\ a_2 c_1 & -a_2 b_1 \end{vmatrix}$$

$$= a_1^2 (b_2 c_3 - b_3 c_2) + a_1 b_1 (-c_3 a_2 + a_3 c_2) + a_1 c_1 (-a_3 b_2 + a_2 b_3) + c_1 b_1 (a_3 a_2 - a_2 a_3) = a_1 \Delta.$$

85.(B) Fundamental concept

$$86.(C) \quad \text{It has a non-zero solution if } \begin{vmatrix} 1 & k & -1 \\ 3 & -k & -1 \\ 1 & -3 & 1 \end{vmatrix} = 0 \Rightarrow -6k + 6 = 0 \Rightarrow k = 1.$$

87.(D) For the system of given homogeneous equations $\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -1 & -1 \\ 1 & -3 & 1 \end{vmatrix} = 1(-1-3) - 1(3+1) - 1(-9+1)$

$$= -4 - 4 + 8 = 0$$

$\therefore$ There are infinite number of solutions.

88.(D) The given system of homogeneous equations has a non-zero solution if, $\Delta = 0$

i.e., $\begin{vmatrix} 1 & 1 & -1 \\ 3 & -\alpha & -3 \\ 1 & -3 & 1 \end{vmatrix} = -2\alpha - 6 = 0$, i.e. if $\alpha = -3$.

89.(B) $\frac{d^n}{dx^n}[\Delta(x)] = \begin{vmatrix} \frac{d^n}{dx^n} x^n & \frac{d^n}{dx^n} \sin x & \frac{d^n}{dx^n} \cos x \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix} = \begin{vmatrix} n! & \sin\left(x + \frac{n\pi}{2}\right) & \cos\left(x + \frac{n\pi}{2}\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix}$

$$\Rightarrow \left(\Delta^n(x)\right)_{x=0} = \begin{vmatrix} n! & \sin\left(0 + \frac{n\pi}{2}\right) & \cos\left(0 + \frac{n\pi}{2}\right) \\ n! & \sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) \\ a & a^2 & a^3 \end{vmatrix} = 0. \text{ \{Since } R_1 \equiv R_2 \text{ \}}$$

90.(D) Let $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $|A| = 3(-7) - 5(3) + 7(5) = -1$

$$\text{Adj}(A) = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 5 & -19 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|}$$

$$A^{-1} = \begin{bmatrix} 7 & 3 & -26 \\ 3 & 1 & -11 \\ -5 & -5 & 19 \end{bmatrix}$$

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Level – 2

91.(AD) $(B^T AB)^T = B^T A^T (B^T)^T = B^T A^T B = B^T AB$ if A is symmetric.

Therefore, $B^T AB$ is symmetric if A is symmetric.

Also, $(B^T AB)^T = B^T A^T B = B^T (-A)B = -(B^T A^T B)$

Therefore, $B^T AB$ is skew-symmetric if A is skew-symmetric.

92.(ABC) We have, $|A(\theta)| = 1$

Hence, A is invertible.

$$A(\pi + \theta) = \begin{bmatrix} \sin(\pi + \theta) & -\cos(\pi + \theta) \\ i \cos(\pi + \theta) & \sin(\pi + \theta) \end{bmatrix} = \begin{bmatrix} -\sin \theta & -i \cos \theta \\ -i \cos \theta & -\sin \theta \end{bmatrix} = -A(\theta)$$

$$\text{adj}(A(\theta)) = \begin{bmatrix} \sin \theta & -i \cos \theta \\ -i \cos \theta & \sin \theta \end{bmatrix} \Rightarrow A(\theta)^{-1} = \begin{bmatrix} \sin \theta & -i \cos \theta \\ -i \cos \theta & \sin \theta \end{bmatrix} = A(\pi - \theta)$$

93.(ABC) $\therefore |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3 + 4) + 3(2 - 0) + 4(-2 - 0) = 1$

$\therefore \text{adj}(\text{adj } A) = |A|^{3-2} A = A$ and $|\text{adj}(\text{adj } A)| = |A| = 1$

Also, $|\text{adj } A| = |A|^{3-1} = |A|^2 = 1^2 = 1$

94.(ABC) $|A^{-1}| = -1 \Rightarrow |A| = -1$

Now, use $\text{adj } A = |A| A^{-1}$ and $A = (A^{-1})^{-1}$

95.(BC) $(-A)^{-1} = \frac{\text{adj}(-A)}{|-A|} = \frac{(-1)^{n-1} \text{adj}(A)}{(-1)^n |A|} = \frac{\text{adj}(A)}{-|A|} = -A^{-1}$ (for any value of n)

Given, $A^n = O$

Now, $(I - A)(I + A + A^2 + \dots + A^{n-1}) = I - A^n = I$

or $(I - A)^{-1} = I + A + A^2 + \dots + A^{n-1}$

96.(ABC) If $|A| \neq 0$, then

$AB = AC$ or $A^{-1}AB = A^{-1}AC$ or $B = C$

Also, if A is orthogonal matrix, then, $AA^T = I$

$\Rightarrow [AA^T] = 1 \Rightarrow |A|^2 = 1 \Rightarrow A$ is invertible

97.(ABC) Applying $R_3 \rightarrow R_3 - R_2$, $R_2 \rightarrow R_2 - R_1$, we get $|a| = 3 \begin{vmatrix} a_1 & a_2 & a_3 \\ d & d & d \\ d & d & d \end{vmatrix} = 0$

where d is the common difference of the A.P.

Therefore, the given system of equations has infinite number of solutions. Also, $|B| = a_1^2 + a_2^2 \neq 0$

$$98.(BC) \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{or} \quad (A^{-1})^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\text{Similarly, } (A^{-1})^3 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\text{and} \quad (A^{-1})^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} A^{-n} = \lim_{n \rightarrow \infty} \begin{bmatrix} 1/n & 0 \\ -1 & 1/n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} A^{-n} = \lim_{n \rightarrow \infty} \begin{bmatrix} 1/n^2 & 0 \\ -1/n & 1/n^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

99.(AD) Here X is a $n \times 1$ matrix, C is a $n \times n$ matrix, and X^T is a $1 \times n$ matrix. Hence, $X^T C X$ is a 1×1 matrix. Let $X^T C X = k$. Then, $(X^T C X)^T = X^T C^T (X^T)^T = X^T (-C) X = -X^T C X = -k$
 $\Rightarrow k = -k \Rightarrow k = 0$
 $\Rightarrow X^T C X$ is a null matrix

$$100.(ABC) \quad S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{or} \quad S^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\text{We have, } SA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} b+c & c+a & b-c \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix} = \begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix}$$

$$\therefore SAS^{-1} = \begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix} = \text{diag}(2a, 2b, 2c)$$

101.(ABCD) We have, $A^2 B = A(AB) = AA = A^2$, $B^2 A = B(BA) = BB = B^2$,
 $ABA = A(BA) = AB = A$ and $BAB = B(AB) = BA = B$.

102.(4) $|A| = (2k+1)^3$, $|B| = 0$ (since B is skew-symmetric of order 3)

$$\Rightarrow \det(\text{adj } A) = |A|^{n-1} = ((2k+1)^3)^2 = 10^6$$

$$\Rightarrow 2k+1 = 10 \text{ or } 2k = 9$$

$$\Rightarrow [K] = 4$$

$$103.(BD) \quad a_{ij} = \begin{cases} x; & \text{if } i = j, x \in R \\ 1; & \text{if } |i - j| = 1 \\ 0; & \text{otherwise} \end{cases} \Rightarrow A = \begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix} \Rightarrow |A| = x^3 - 2x$$

$$\text{If } f(x) = x^3 - 2x \Rightarrow f'(x) = 3x^2 - 2 = (\sqrt{3}x - \sqrt{2})(\sqrt{3}x + \sqrt{2})$$

$$\text{So } x = \frac{\sqrt{2}}{\sqrt{3}} \text{ point of minima and } x = \frac{-\sqrt{2}}{\sqrt{3}} \text{ is maxima.}$$

104.(C) Given $MN = NM$

$$\begin{aligned} & M^2 N^2 (M^T N)^{-1} (MN^{-1})^T \\ &= M^2 N^2 N^{-1} (M^T)^{-1} (N^{-1})^T M^T = M^2 N (M^T)^{-1} (N^{-1})^T M^T = -M^2 N (M)^{-1} (N^T)^{-1} M^T \\ &= +M^2 NM^{-1} N^{-1} M^T = -MNMM^{-1} N^{-1} M = -MNN^{-1} M = -M^2 \end{aligned}$$

105.(C,D) (A) $(N^T MN)^T = N^T M^T N = N^T MN$ if is symmetric and is $-N^T MN$ if M is skew symmetric

(B) $(MN - NM)^T = N^T M^T - M^T N^T = NM - MN = -(MN - NM)$. So, $(MN - NM)$ is skew symmetric

(C) $(MN)^T = N^T M^T = NM \neq MN$ if M and N symmetric. so, MN is not symmetric

(D) $(adj.M)(adj.N) = (adj.NM) \neq adj(MN)$.

106.(4) Given that $AA^T = 4I \Rightarrow |A|^2 = 4$

Or $|A| = \pm 2$

$$\text{So } A^T = 4A^{-1} = 4 \frac{adj A}{|A|} \Rightarrow \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \frac{4}{|A|} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{21} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

$$\text{Now } a_{ij} = \frac{4}{|A|} c_{ij} \Rightarrow -2c_{ij} = \frac{4}{|A|} c_{ij} \text{ (as } a_{ij} + 2c_{ij} = 0)$$

$$\text{Now } [A + 4I] = [A + AA^T] = |A| [I + A^T] = -2 [(I + A)^T] = -2 [I + A]$$

$$\Rightarrow |A + 4I| + 2|A + I| = 0$$

So on comparing, we get $5\lambda = 2 \Rightarrow \lambda = \frac{2}{5}$. Hence, $10\lambda = 4$

107.(A,D) $(B^T AB)^T = B^T A^T (B^T)^T = B^T A^T B = B^T AB$ if A is symmetric.

There fore, $B^T AB$ is symmetric if A is symmetric. Also $(B^T AB^T) = B^T A^T B = B^T (-A) B = -(B^T A^T B)$

Therefore $B^T AB$ if A is skew -symmetric if A is skew- symmetric.

108. (i) $(A+B)C = (A+B)(A+B)^{-1}(A-B)$

$$\Rightarrow (A+B)C = A-B \quad (1)$$

$$\begin{aligned} C^T &= (A-B)^T ((A+B)^{-1})^T = (A+B) ((A+B)^T)^{-1} \left\{ \text{as } |A+B| \neq 0 \Rightarrow |(A+B)^T| \neq 0 \Rightarrow |A-B| \neq 0 \right\} \\ &= (A+B)(A-B)^{-1} \quad (2) \end{aligned}$$

Form (1) and (2), we get

$$\begin{aligned} C^t (A+B)C &= (A+B)(A-B)^{-1}(A-B) \\ &= (A+B) \quad (3) \end{aligned}$$

(II) Taking transpose in (3), we get

$$C^T (A+B)^T (C^T)^T = (A+B)^T$$

$$C^T (A-B)C = A-B \quad (4)$$

(III) Adding (3) and (4), we get

$$C^T [A+B+A-B]C = 2A$$

$$C^T AC = A$$

109.(B) Note $AB = BA$ $C = (A^{-1} B^{-1})^5 I^5 I$
 $|C| = 1$

110.(BC) $A^2 = \begin{vmatrix} a & b & 0 & a & b & 0 \\ c & 0 & b & c & 0 & b \\ 0 & c & a & 0 & c & a \\ a^2 & bc & ab & b^2 & a & b \\ ac & 2bc & ab & c & 0 & b \\ c^2 & ac & a^2 & bc & 0 & c \end{vmatrix}$
 $A^3 = \begin{vmatrix} a^2 & bc & ab & b^2 & a & b \\ ac & 2bc & ab & c & 0 & b \\ c^2 & ac & a^2 & bc & 0 & c \\ a^3 & 2abc & a^2b & 2b^2c & 0 & 0 \\ a^2c & 2bc^2 & 0 & 2b^2c & a^2b & a^2 \\ 0 & 2bc^2 & a^2c & 2abc & a^3 & 0 \end{vmatrix}$
 $A^9 = (a^2 - 2bc)^3 A^3 = (a^2 - 2bc)^4 A$
 $A(A^2 - (2bc - a^2)I) = 0 \quad |A^2 - (2bc - a^2)I| = 0 \text{ else } A = 0$

111.(AB) $|A| = 0, A' = A$
 $|B| = 0, B' = B$
 $AB = BA$
 $A^3 B^3 (B'A)^{-1} (A^{-1} B^{-1})' AB = A^3 B^3 (A^{-1} B^{-1}) ((BA)')^{-1} AB = A^3 B^3 A^{-1} B^{-1} (AB)^{-1} AB = A^2 B^2 = B^2 A^2$

112.(BC) $P^T = \lambda P - \mu I$
 $P = \lambda P^T - \mu I = \lambda(\lambda P - \mu I) - \mu I$
 $P = \lambda^2 P - (\lambda\mu - \mu)I \quad P = \frac{\mu(\lambda - 1)I}{1 - \lambda^2} = \frac{\mu}{1 - \lambda} I, \text{ Trace of } P = \frac{3\mu}{1 - \lambda}$

113.(AB) $M^3 = N^6$
 $M^3 - N^6 = 0 \quad (M - N^2)(M^2 + MN^2 + N^4) = 0$
 $|M^2 + MN^2 + N^4| = 0, \text{ else } M = N^2$
 $|M^2 N^2 - M^3 - MN^4| = |M^2 N^2 - N^6 - MN^4| = |N|^2 |M^2 - N^4 - MN^2| = 0$
If $(M^2 N^2 - M^3 - MN^4) = 0 \& |M^2 N^2 - M^3 - MN^4| = 0$
System have non trivial or non zero solution.

114.(BC) $D = \begin{vmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 6 & 5 & \lambda \end{vmatrix}$
 $3(2\lambda - 15) - (\lambda - 18) - 4(5 - 12) - 7\lambda - 35 = 0 \text{ for } \lambda = 5$
 $3x - y + 4z = 3, \quad x + 2y + 3z = 2$

$$\text{Let } z = 0 \quad x = \frac{4}{7}, y = \frac{9}{7} \quad \frac{7x-4}{5} = \frac{7y-9}{13} = \frac{z-0}{1}$$

$$z = 1 \quad x = \frac{1}{7}, y = \frac{4}{7} \quad \frac{7x-1}{5} = \frac{7y-4}{13} = \frac{3-1}{1}$$

115.(2) $abc = 1$

$$\begin{pmatrix} 2a & b & c \\ b & 2c & a \\ c & a & 2b \end{pmatrix} \begin{pmatrix} 2a & b & c \\ b & 2c & a \\ c & a & 2b \end{pmatrix} = \begin{pmatrix} 4a^2+b^2+c^2 & 2ab+2bc+ac & 2ac+ab+2bc \\ 2ab+2bc+ac & b^2+4c^2+a^2 & bc+2ca+2ab \\ 2ac+ab+2bc & cb+2ac+2ab & c^2+a^2+4b^2 \end{pmatrix} = 4^{1/3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$4a^2+b^2+c^2 = a^2+b^2+4c^2 = a^2+4b^2+c^2 = 4^{1/3}$$

$$ab+bc+ca = 0$$

$$a^2+b^2+c^2 = 2^{-1/3} \Rightarrow (a+b+c)^2 = 2^{-1/3}$$

$$|A| = 10 - 2(a^3+b^3+c^3) > 0 \Rightarrow a^3+b^3+c^3 < 5$$

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = \pm 2^{-1/6} \cdot 2^{-1/3} = \pm 2^{-1/2}$$

116.(0) $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$$\Rightarrow A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^4 = A^2 \cdot A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$\Rightarrow A^8 = \begin{bmatrix} 3^4 & 0 \\ 0 & 3^4 \end{bmatrix} \quad \text{and} \quad A^6 = A^4 \cdot A^2 = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

Let $V = \begin{bmatrix} x \\ y \end{bmatrix}$

$$A^8 + A^6 + A^4 + A^2 + I$$

$$\begin{bmatrix} 81 & 0 \\ 0 & 81 \end{bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}$$

$$(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 121x \\ 121y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$$\Rightarrow x = 0 \text{ and } y = 1/11 \Rightarrow V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1/11 \end{bmatrix}$$

117.(4) $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix.

$$\Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$$

$$\text{or } \begin{bmatrix} a^2 + bc & ab + b - ab \\ ac + c - ac & bc + (1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a^2 + bc & b \\ c & bc + (1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$$

$$\begin{aligned} \text{or } a^2 + bc &= a \\ a - a^2 &= bc = 14 \quad (\text{given}) \\ f(a) &= 1/4 \end{aligned}$$

$$118.(2) \quad A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$\Rightarrow A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^2 = I \Rightarrow A^4 = A^6 = A^8 = \dots = I$$

$$\text{Now } A^x = I$$

$$\Rightarrow x = 2, 4, 6, 8, \dots$$

$$\begin{aligned} \therefore \Sigma(\cos^x \theta + \sin^x \theta) &= (\cos^2 \theta + \sin^2 \theta) + (\cos^4 \theta + \sin^4 \theta) + (\cos^6 \theta + \sin^6 \theta) + \dots \\ &= (\cos^2 \theta + \cos^4 \theta + \cos^6 \theta + \dots) + (\sin^4 \theta + \sin^6 \theta + \dots) \\ &= \frac{\cos^2 \theta}{1 - \cos^2 \theta} + \frac{\sin^2 \theta}{1 - \sin^2 \theta} = \cot^2 \theta + \tan^2 \theta \end{aligned}$$

which has minimum value 2

$$119.(1) \quad A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$\text{Hence, det. } A = \sec^2 x$$

$$\therefore \det A^T = \sec^2 x$$

$$\text{Now } f(x) = \det(A^T A^{-1}) = (\det A^T)(\det A^{-1}) = (\det A^T)(\det A)^{-1}$$

$$= (\det A^T)(\det A)^{-1} = \frac{\det(A^T)}{\det(A)} = 1$$

$$\text{Hence, } f(x) = 1$$

$$120.(9) \quad \text{Given } A^2 = A$$

$$\Rightarrow I = (I - 0.4A)(I - \alpha A)$$

$$= I - I\alpha A - 0.4AI + 0.4\alpha A^2 = I - A\alpha - 0.4A + 0.4\alpha A = I - A(0.4 + \alpha) + 0.4\alpha A$$

$$\Rightarrow 0.4\alpha = 0.4 + \alpha \Rightarrow \alpha = -2/3 \Rightarrow |9\alpha| = 6$$

121.(7) We have $AB = \begin{bmatrix} 3ax^2 & 3bx^2 & 3cx^2 \\ a & b & c \\ 6ax & 6bx & 6cx \end{bmatrix}$

Now $tr.(AB) = tr.(C)$

$$\Rightarrow 3ax^2 + b + 6cx = (x+2)^2 + 2x + 5x^2 \quad \forall x \in R \text{ (Identity)}$$

$$\Rightarrow 3ax^2 + 6cx + b = 6x^2 + 6x + 4 \Rightarrow a = 2, c = 1, b = 4$$

122.(4) Give that $AA^T = 4I$

$$\Rightarrow |A|^2 = 4 \text{ or } |A| = \pm 2$$

So $A^T = 4A^{-1} = 4 \frac{adj A}{|A|} \Rightarrow \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \frac{4}{|A|} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$

Now $a_{ij} = \frac{4}{|A|} c_{ij} \Rightarrow -2c_{ij} = \frac{4}{|A|} c_{ij} \text{ (as } a_{ij} + 2c_{ij} = 0) \Rightarrow |A| = -2$

Now $|A + 4I| = |A + AA^T| = |A| |I + A^T| = -2 |(I + A)^T| = -2 |I + A|$

$$\Rightarrow |A + 4I| + 2|A + I| = 0$$

So on comparing, we get $5\lambda = 2 \Rightarrow \lambda = \frac{2}{5}$

Hence, $10\lambda = 4$

123.(0) For idempotent matrix, $A^2 = A$

$$\Rightarrow A^{-1}A^2 = A^{-1}A \quad (\because A \text{ is nonsingular})$$

or $A = I$

Thus, nonsingular idempotent matrix is always a unit matrix.

$$\therefore I^2 - 3I = 1 \Rightarrow I = \pm 2$$

$$m^2 - 8 = 1 \Rightarrow m = \pm 3$$

$$n^2 - 15 = 1 \Rightarrow n = \pm 4 \quad \text{and} \quad p = q = r = 0$$

Hence, the required sum is 0.

124.(3) Equation $x^3 + ax^2 + bx + c = 0$ has roots α, β, γ . Therefore,

$$\alpha + \beta + \gamma = -a$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = b$$

Since the given system of equations has non-trivial solutions,

We have

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$$

or $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = 0$ or $(\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha] = 0$

or $(\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)] = 0 \Rightarrow -a[a^2 - 3b] = 0$ or $a^2/b = 3$

125.(2) We have $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$

Since $a_n = \frac{20}{n}; d = \frac{1}{20}$

Hence, $D = \begin{vmatrix} 20 & \frac{20}{2} & \frac{20}{3} \\ \frac{20}{4} & \frac{20}{5} & \frac{20}{6} \\ \frac{20}{7} & \frac{20}{8} & \frac{20}{9} \end{vmatrix} = \frac{(20)^3}{4 \times 7} \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{4}{5} & \frac{2}{3} \\ 1 & \frac{7}{8} & \frac{9}{9} \end{vmatrix}$

$R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$= \frac{(20)^3}{4 \times 7} \begin{vmatrix} 0 & -\frac{3}{10} & -\frac{1}{3} \\ 0 & -\frac{3}{40} & -\frac{1}{9} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix} = \frac{50}{21} \Rightarrow [D] = 2$

126.(8) Let $D = \begin{vmatrix} (\beta + \gamma - \alpha - \delta)^4 & (\beta + \gamma - \alpha - \delta)^2 & 1 \\ (\gamma + \alpha - \beta - \delta)^4 & (\gamma + \alpha - \beta - \delta)^2 & 1 \\ (\alpha + \beta - \gamma - \delta)^4 & (\alpha + \beta - \gamma - \delta)^2 & 1 \end{vmatrix}$

Applying $R_1 \rightarrow R_1 - R_3$

$R_2 \rightarrow R_2 - R_3$

$= \begin{vmatrix} (\beta + \gamma - \alpha - \delta)^4 - (\alpha + \beta - \gamma - \delta)^4 & (\beta + \gamma - \alpha - \delta)^2 - (\alpha + \beta - \gamma - \delta)^2 & 0 \\ (\gamma + \alpha - \beta - \delta)^4 - (\alpha + \beta - \gamma - \delta)^4 & (\gamma + \alpha - \beta - \delta)^2 - (\alpha + \beta - \gamma - \delta)^2 & 0 \\ (\alpha + \beta - \gamma - \delta)^4 & (\alpha + \beta - \gamma - \delta)^2 & 1 \end{vmatrix}$

$= 4(\beta - \delta)(\gamma - \alpha), 4(\alpha - \delta)(\gamma - \beta)$

$\times \begin{vmatrix} (\beta + \gamma - \alpha - \delta)^2 + (\alpha + \beta - \gamma - \delta)^2 & 1 & 0 \\ (\gamma + \alpha - \beta - \delta)^2 + (\alpha + \beta - \gamma - \delta)^2 & 1 & 0 \\ (\alpha + \beta - \gamma - \delta)^4 & (\alpha + \beta - \gamma - \delta)^2 & 1 \end{vmatrix}$

Applying $R_1 \rightarrow R_1 - R_2$

$= 16(\beta - \delta)(\gamma - \alpha)(\alpha - \delta) \cdot 4(\gamma - \delta)(\beta - \alpha)$

$\times \begin{vmatrix} 1 & 0 & 0 \\ (\gamma + \alpha - \beta - \delta)^2 + (\alpha + \beta - \gamma - \delta)^2 & 1 & 0 \\ (\alpha + \beta - \gamma - \delta)^4 & (\alpha + \beta - \gamma - \delta)^2 & 1 \end{vmatrix}$

$= -64(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)(\gamma - \delta)$

$$127.(4) \Delta = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$$

Applying $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$

$$\Delta = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ x+1 & x+1 & x+1 \\ x+2 & 2(x+2) & 6(x+2) \end{vmatrix} = 0$$

$$\therefore \Delta = (x+1)(x+2) \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 0$$

$$\therefore \Delta = (x+1)(x+2)[(x+2).4 - (2x+3).5 + (3x+4).1] = 0$$

$$\Delta = (x+1)(x+2)(-3x-3) = 0 \quad \text{or} \quad (x+1)^2(x+2) = 0$$

$$\therefore x = -1, -1, -2$$

$$128.(0) \Delta' = \begin{vmatrix} A_3'(x) & (xA_2(x))' & (x^2A_1(x))' \\ A_3''(x) & (xA_2(x))'' & (x^2A_1(x))'' \\ A_3^{iv}(x) & (xA_2(x))^{iv} & (x^2A_1(x))^{iv} \end{vmatrix} = 0$$

$$129. \Delta = \begin{vmatrix} 1 & \cos \alpha \cos \beta + \sin \alpha \sin \beta & \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \\ \cos \alpha \cos \beta + \sin \alpha \sin \beta & 1 & \cos \gamma \cos \beta + \sin \gamma \sin \beta \\ \cos \alpha \cos \gamma + \sin \alpha \sin \gamma & \cos \beta \cos \gamma + \sin \alpha \sin \gamma & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos \alpha & \sin \alpha & 0 \\ \cos \beta & \sin \beta & 0 \\ \cos \gamma & \sin \gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos \alpha & \sin \alpha & 0 \\ \cos \beta & \sin \beta & 0 \\ \cos \gamma & \sin \gamma & 0 \end{vmatrix} = 0 \times 0 = 0$$

$$130. \Delta = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} a^2 - 2ax + x^2 & b^2 - 2bx + x^2 & c^2 - 2cx + x^2 \\ a^2 - 2ay + y^2 & b^2 - 2by + y^2 & c^2 - 2cy + y^2 \\ a^2 - 2az + z^2 & b^2 - 2bz + z^2 & c^2 - 2cz + z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix} = - \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} a^2 & 2a & 1 \\ b^2 & 2b & 1 \\ c^2 & 2c & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \quad (\text{in second determinant } C_1 \leftrightarrow C_2)$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} = 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$$

$$\text{Multiplying row by row, we get } \Delta = \begin{vmatrix} 1+2ax+a^2x^2 & 1+2bx+b^2x^2 & 1+2cx+c^2x^2 \\ 1+2ay+a^2y^2 & 1+2by+b^2y^2 & 1+2cy+c^2y^2 \\ 1+2az+a^2z^2 & 1+2bz+b^2z^2 & 1+2cz+c^2z^2 \end{vmatrix}$$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$$

131. Keeping in mind the term $2bc - a^2$, we have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = \Delta$ [row by row multiplication]

Therefore,

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = [a(bc - a^2) + b(ac - b^2) + c(ab - c^2)]^2 = [3abc - a^3 - b^3 - c^3]^2$$

132. Rewriting the given determinant, we have $\Delta = \frac{1}{c} \begin{vmatrix} ah+bg & gc & ab+ch \\ bf+ba & fc & hb+bc \\ af+bc & c^2 & bg+fc \end{vmatrix} = -\frac{a}{c} \begin{vmatrix} ah+bg & a & ch \\ bf+ba & h & bc \\ af+bc & g & fc \end{vmatrix}$

By operating in second determinant $C_3 \rightarrow C_3 + bC_2$, we get $\Delta = \frac{1}{c} \Delta_1 - \frac{a}{c} \begin{vmatrix} ah+bg & a & ab+ch \\ bf+ba & h & hb+bc \\ af+bg & g & bg+fc \end{vmatrix}$

$$= \frac{1}{c} \Delta_1 + \frac{1}{c} \begin{vmatrix} ah+bg & -a^2 & ab+ch \\ bf+ba & -ah & hb+bc \\ af+bc & -ag & bg+fc \end{vmatrix} = \frac{1}{c} \begin{vmatrix} ah+bg & gc-a^2 & ab+ch \\ bf+ba & fc-ah & hb+bc \\ af+bc & c^2-ag & bg+fc \end{vmatrix} = -\frac{1}{c} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \times \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

133. $\frac{dy}{dx} = \begin{vmatrix} \cos x & -\sin x & \cos x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \sin x & \cos x & \sin x \\ -\sin x & -\cos x & -\sin x \\ x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ 1 & 0 & 0 \end{vmatrix}$

$$= 0 - \begin{vmatrix} \sin x & \cos x & \sin x \\ \sin x & \cos x & \sin x \\ x & 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 0 + (\cos^2 x + \sin^2 x)$$

134. $\frac{d^n}{dx^n} [f(x)] = \begin{vmatrix} \frac{d^n}{dx^n}(x^n) & n! & 2 \\ \frac{d^n}{dx^n}(\cos x) & \cos \frac{n\pi}{2} & 4 \\ \frac{d^n}{dx^n}(\sin x) & \sin \frac{n\pi}{2} & 8 \end{vmatrix} = \begin{vmatrix} n! & n! & 2 \\ \cos\left(x + \frac{n\pi}{2}\right) & \cos \frac{n\pi}{2} & 4 \\ \sin\left(x + \frac{n\pi}{2}\right) & \sin \frac{n\pi}{2} & 8 \end{vmatrix} \quad (n \in \mathbb{Z})$

$$\Rightarrow \frac{d^n}{dx^n} [f(x)]_x=0 = 0$$

135. $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is formed by the cofactors of matrix $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ -c & -a & -b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a^2 & c^2 & 2ca-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}$$

136. The matrix of rotation will be $\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

Since each side of the square is x, we have

$$x^2 + x^2 = (2\sqrt{2})^2 \quad \text{Or} \quad x = 2 \text{ units}$$

Therefore, the coordinates of the vertices O, A, B, C are (0,0), (2,0), (2,2), (0,2), respectively.

Let after rotation A, B, C map into A', B', C', respectively, while O maps into itself. Then,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

$$\therefore A(2,0) \rightarrow A'(1, \sqrt{3})$$

$$\text{Similarly, } \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3} \\ \sqrt{3}+1 \end{bmatrix} \quad \therefore B(2,2) \rightarrow B'(1-\sqrt{3}, \sqrt{3}+1)$$

$$\text{And } \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \quad \therefore C(0,2) \rightarrow C'(-\sqrt{3}, 1)$$

137.(AD) $a^{10} \quad 1 \quad 0 \quad (a^5 - 1)(a^5 - 1) \quad 0, \quad a^{15} \quad 1 \quad 0 \quad (a^5 - 1)(a^{10} - a^5 - 1) \quad 0$

Common roots are roots of $a^5 - 1$

x, y, z have real part positive

$$\text{Let } x = 1, y = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}, z = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$$

$$\omega = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}, \quad (x-y)(y-z)(z-x)^2$$

$$1 - \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} \cdot 2i \sin \frac{2\pi}{5} \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} \cdot 1^2 - 4 \sin^2 \frac{2\pi}{5} \cdot 1 - \cos \frac{2\pi}{5} + \sin^2 \frac{2\pi}{5}$$

138. Determinant of coefficients is

$$\Delta = \begin{vmatrix} 2 \sin A \cos A & \sin(A+B) & \sin(A+C) \\ \sin(A+B) & 2 \sin B \cos B & \sin(B+C) \\ \sin(A+C) & \sin(B+C) & 2 \sin C \cos C \end{vmatrix}$$

$$\begin{aligned}
 &= \begin{vmatrix} \sin A \cos A + \sin A \cos A & \sin A \cos B + \sin B \cos A & \sin A \cos C + \sin C \cos A \\ \sin A \cos B + \sin B \cos A & 2 \sin B \cos B & \sin B \cos C + \sin C \cos B \\ \sin A \cos C + \sin C \cos A & \sin B \cos C + \sin C \cos B & 2 \sin C \cos C \end{vmatrix} \\
 &= \begin{vmatrix} \sin A \cos A + \sin A \cos A & \sin A \cos B + \sin B \cos A & \sin A \cos C + \sin C \cos A \\ \sin A \cos B + \sin B \cos A & \sin B \cos B + \sin B \cos B & \sin B \cos C + \sin C \cos B \\ \sin A \cos C + \sin C \cos A & \sin B \cos C + \sin C \cos B & \sin C \cos C + \sin C \cos C \end{vmatrix} \\
 &= \begin{vmatrix} \sin A & \cos A & 0 \\ \sin B & \cos B & 0 \\ \sin C & \cos C & 0 \end{vmatrix} \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} = 0
 \end{aligned}$$

139. Let $D = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$\therefore D^2 = D \times D$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} ax_1 & by_1 & cz_1 \\ ax_2 & by_2 & cz_2 \\ ax_3 & by_3 & cz_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} ax_1^2 + by_1^2 + cz_1^2 & ax_1x_2 + by_1y_2 + cz_1z_2 & ax_1x_3 + by_1y_3 + cz_1z_3 \\ ax_1x_2 + by_1y_2 + cz_1z_2 & ax_2^2 + by_2^2 + cz_2^2 & ax_2x_3 + by_2y_3 + cz_2z_3 \\ ax_1x_3 + by_1y_3 + cz_1z_3 & ax_2x_3 + by_2y_3 + cz_2z_3 & ax_3^2 + by_3^2 + cz_3^2 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} d & f & f \\ f & d & f \\ f & f & d \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} d+2f & f & f \\ d+2f & d & f \\ d+2f & f & d \end{vmatrix} \quad [C_1 \rightarrow C_1 + C_2 + C_3]$$

$$= \frac{(d+2f)}{abc} \begin{vmatrix} 1 & f & f \\ 1 & d & f \\ 1 & f & d \end{vmatrix}$$

$$= \frac{(d+2f)}{abc} \begin{vmatrix} 1 & f & f \\ 0 & d-f & 0 \\ 1 & f & d \end{vmatrix}$$

$$= \frac{(d+2f)}{abc} (d-f) \begin{vmatrix} 1 & f \\ 1 & d \end{vmatrix} \quad [R_2 \rightarrow R_2 - R_1]$$

$$= \frac{(d+2f)}{abc} (d-f)(d-f) = \frac{(d+2f)(d-f)^2}{abc}$$

$$\Rightarrow D = (d-f) \left\{ \frac{d+2f}{abc} \right\}^{1/2} = \text{R.H.S}$$

$$\begin{aligned}
 140. \quad \Delta &= \begin{vmatrix} a_1b_1+b_1a_1 & a_1b_2+a_2b_1 & a_1b_3+b_1a_3 \\ a_2b_1+b_2a_1 & a_2b_2+a_2b_2 & a_2b_3+a_3b_2 \\ a_3b_1+b_3a_1 & a_3b_2+b_3a_2 & a_3b_3+a_3b_3 \end{vmatrix} \\
 &= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix} = 0 \quad (1) \\
 &\quad \text{(row by row multiplication)}
 \end{aligned}$$

Now, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$= (lx + my + n)(l'x + m'y + n') = ll'x^2 + (lm' + ml')xy + mm'y^2 + (ln' + l'n)x + (mn' + m'n)y + nn'$$

Comparing the coefficients, we get

$$a = ll', h = \frac{1}{2}(lm' + ml'), b = mm', g = \frac{1}{2}(ln' + l'n), f = \frac{1}{2}(mn' + m'n), c = nn'$$

$$\begin{aligned}
 \therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} &= \begin{vmatrix} ll' & \frac{1}{2}(lm' + ml') & \frac{1}{2}(ln' + l'n) \\ \frac{1}{2}(lm' + ml') & mm' & \frac{1}{2}(mn' + m'n) \\ \frac{1}{2}(ln' + l'n) & \frac{1}{2}(mn' + m'n) & nn' \end{vmatrix} \\
 &= \frac{1}{8} \begin{vmatrix} 2ll' & lm' + l'm & ln' + l'n \\ lm' + l'm & 2mm' & mn' + m'n \\ ln' + l'n & mn' + m'n & 2nn' \end{vmatrix} \\
 &= 0 \quad \text{[From (1)]}
 \end{aligned}$$

141.(2) Applying in $C_1 \rightarrow C_1 + C_2 + C_3$, then

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} \quad (\because a^2 + b^2 + c^2 + 2 = 0)$$

Now, applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & (1-x) \end{vmatrix} = (1-x)^2$$

Hence, degree of $f(x) = 2$.

142.(841)

$$\begin{vmatrix} 3 & -2 & 1 \\ \lambda & -14 & 15 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 3(-42 - 30) + 2(3\lambda - 15) + 1(2\lambda + 14) = 0 \Rightarrow 8\lambda = 232$$

$$\therefore \lambda = 29 \Rightarrow \lambda^2 = 29^2 = 841$$

143.(2) Differentiating w.r.t. x , we get

$$\begin{aligned} -f'(x) &= \begin{vmatrix} -\sin x & x & 1 \\ 2\cos x & x^2 & 2x \\ \sec^2 x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & 1 & 1 \\ 2\sin x & 2x & 2x \\ \tan x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 0 \\ 2\sin x & x^2 & 2 \\ \tan x & x & 0 \end{vmatrix} \\ \Rightarrow -\frac{f'(x)}{x} &= \begin{vmatrix} -\sin x & 1 & 1 \\ 2\cos x & x & 2x \\ \sec^2 x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & 1 & 0 \\ 2\sin x & x & 2 \\ \tan x & 1 & 0 \end{vmatrix} \quad (\because \text{the second determinant is zero}) \\ \Rightarrow -\lim_{x \rightarrow 0} \frac{f'(x)}{x} &= \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2 \end{aligned}$$

144.(1) Applying $C_1 \rightarrow C_1 - 2\sin x C_3$ and $C_2 \rightarrow C_2 + 2\cos x C_3$, we obtain

$$\begin{aligned} f(x) &= \begin{vmatrix} 2 & 0 & -\sin x \\ 0 & 2 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix} \\ &= 2\sin^2 x + 2\cos^2 x = 2 \\ \Rightarrow f'(x) &= 0 \\ \therefore \int_0^{\pi/2} [f(x) + f'(x)] dx &= \int_0^{\pi/2} 2 dx = 2 \left(\frac{\pi}{2} \right) = \pi \end{aligned}$$

145.(3) Let $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

[Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$]

$$= (a+b+c) \begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix}$$

[Expanding along C_1]

$$\begin{aligned} &= (a+b+c) \{(2b+a)(2c+a) - (-b+a)(-c+a)\} \\ &= (a+b+c) \{(4bc+2ab+2ca+a^2) - (bc-ab-ac+a^2)\} \\ &= (a+b+c) (3bc+3ab+3ca) \\ &= 3(a+b+c) (ab+bc+ca) \end{aligned}$$

146.(6) Applying $C_3 \rightarrow C_3 - C_2$ and $C_2 \rightarrow C_2 - C_1$, we obtain

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ \phi_1(x_1) & x_2 - x_1 & x_3 - x_2 \\ \phi_2(x_1) & x_2^2 - x_1^2 + b_1(x_2 - x_1) & x_3^2 - x_2^2 + b_1(x_3 - x_2) \end{vmatrix}$$

$$\begin{aligned}
&= (x_2 - x_1)(x_3 - x_2) \begin{vmatrix} 1 & 1 \\ x_2 + x_1 + b_1 & x_3 + x_2 + b_1 \end{vmatrix} \\
&= (x_2 - x_1)(x_3 - x_2)(x_3 - x_1) \\
&= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)
\end{aligned}$$

147.(8) Since we should be able to solve (i) and (ii) for x and y , we must have $ab - h^2 \neq 0$. We can rewrite (iii) as $x(ax + hy + g) + y(hx + by + f) + gx + fy + c + t = 0$.

Since any solution of (i) and (ii) must be a solution of (iii), the given system of equations can be rewritten as

$$ax + hy + g = 0, \quad hx + by + f = 0$$

$$\text{and} \quad gx + fy + c + t = 0$$

Since the above system of equations is consistent, we must have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+t \end{vmatrix} = 0$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} + \begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & t \end{vmatrix} = 0$$

$$\Rightarrow [abc + 2fgh - af^2 - bg^2 - ch^2] + t(ab - h^2) = 0 \quad \Rightarrow \quad t = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab}$$

148.(9) $D_c = \begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix}$ is the determinant formed by the cofactors of the determinant

$$D = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix}$$

$$\text{Hence, } D_c = D^2 = 3^2 = 9$$

149.(1) $A^2 = 3A - 2I$

$$\begin{aligned}
\text{Now, } A^4 &= (A^2)^2 = (3A - 2I)^2 \\
&= 9A^2 - 12A \times I + 4I^2 \\
&= 9A^2 - 12A + 4I \\
&= 9(3A - 2I) - 12A + 4I \\
&= 15A - 14I \\
A^8 &= (A^4)^2 = (15A - 14I)^2 \\
&= 225A^2 - 420A \times I + 196I^2 \\
&= 225(3A - 2I) - 420A + 196I \\
&= 255A - 254I
\end{aligned}$$

150.(2) $|A| = 2$; $|B| = 3$; $|C| = 5$

$$\det(A^2 BC^{-1}) = |A^2 BC^{-1}| = \frac{|A|^2 |B|}{|C|}$$

$$= \frac{4 \times 3}{5} = \frac{12}{5}$$

151.(4) $A^T A = I$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ab + bc + ca \\ ab + bc + ca & a^2 + b^2 + c^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2 + b^2 + c^2 = 1 \quad \dots(i)$$

$$\text{and } ab + bc + ca = 0 \quad \dots(ii)$$

$$\begin{aligned} \text{Now, } a^3 + b^3 + c^3 &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc \\ &= (a + b + c) + 3abc \quad \dots(iii) \end{aligned}$$

$$\begin{aligned} \text{Now, } (a + b + c)^2 &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\ &= 1 + 2 \times 0 = 1 \end{aligned}$$

$$\Rightarrow a + b + c = 1 \quad (\because a, b, c \text{ are real positive numbers})$$

$$\text{Now from equation (iii), } a^3 + b^3 + c^3 = 1 + 3 = 4$$

152.(896)

$$\therefore \sum_{r=0}^7 A_r = \sum_{r=0}^7 \begin{bmatrix} a_r & 0 \\ 0 & b_r \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \sum_{r=0}^7 a_r = \sum_{r=0}^7 r \cdot ({}^7C_r)$$

$$\text{and } b = \sum_{r=0}^7 b_r = \sum_{r=0}^7 (7-r) \cdot ({}^7C_r)$$

$$\Rightarrow a + b = 7 \sum_{r=0}^7 {}^7C_r = 7(2^7) = 7 \times 128 = 896$$

153.(8) Since, the system of equations has infinitely many solution, therefore $D = D_1 = D_2 = D_3 = 0$

$$\text{Here, } D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \alpha \end{vmatrix} = 1(2\alpha - 9) - 1(\alpha - 3) + 1(3 - 2) = \alpha - 5$$

$$\text{and } D_3 = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 9 \\ 1 & 3 & \beta \end{vmatrix} = 1(2\beta - 27) - 1(\beta - 9) + 5(3 - 2) = \beta - 13$$

$$\text{Now, } D = 0$$

$$\Rightarrow \alpha - 5 = 0 \quad \Rightarrow \quad \alpha = 5$$

$$\text{and } D_3 = 0 \quad \Rightarrow \quad \beta - 13 = 0$$

$$\Rightarrow \beta = 13$$

$$\therefore \beta - \alpha = 13 - 5 = 8$$

154.(4.5) Let the matrix be $\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$.

We have five entries 1 and the remaining four entries are 0.

Since, the matrix is symmetric we must have an even number of zeros for $i \neq j$.

Hence, we have two cases

(i) Two entries in the diagonal are zero. We can select two places from three (in diagonal) in ${}^3C_2$ ways.

(ii) Now we have to select elements for the upper triangle.

For the upper triangle, we have three places of which one entry is 0 and two entries are 1.

One place from three can be selected in ${}^3C_1$ ways.

Hence, the number of matrices is ${}^3C_2 \times {}^3C_1 = 9$

155.(1) $x = -\frac{1}{13}, y = \frac{7}{13}$

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1.(C) For non-zero solutions, we have $\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$

$$\Rightarrow 1(-3k + 8) - k(-9 + 4) + 3(12 - 2k) = 0 \text{ which gives } k = 11$$

Now, the system of equations become

$$x + 11y + 3z = 0, \quad 3x + 11y - 2z = 0, \quad 2x + 4y - 3z = 0$$

The equation (i) and (iii) gives $3x + 15y = 0$ i.e. $x = -5y$

Putting $x = -5y$ in (i), we have $-5y + 11y + 3z = 0$

$$\Rightarrow z = -2y \text{ Now } \frac{xz}{y^2} = \frac{(-5y)(-2y)}{y^2} = 10$$

2.(D) As both sides are polynomial in x , let's set $x = 0$ to obtain

$$\begin{vmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{vmatrix} = A^3$$

which gives $A^3 = -64 \therefore A = -4$

taking x common from all rows of given determinant, we get

$$\begin{vmatrix} 1 - \frac{4}{x} & 2 & 2 \\ 2 & 1 - \frac{4}{x} & 2 \\ 2 & 2 & 1 - \frac{4}{x} \end{vmatrix} = \left(B - \frac{4}{x} \right) \left(1 + \frac{4}{x} \right)^2$$

Take the limit as $x \rightarrow \infty$ to obtain $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = B \Rightarrow B = 5$

Alternative solution :

The most efficient way to obtain the result is to use this result

$$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = (a + 2b)(a - b)^2$$

This result gives $A = -4, B = 5$

Note that the determinant, when $a = b$, vanish and all the three rows become identical hence $(a - b)^2$ is a factor.

3.(A) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in R \quad \dots (i)$

According to given condition, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \text{ for some scalar } \lambda.$$

$$\Rightarrow \begin{array}{lll} a = \lambda & \dots \text{ (ii); } & 2c + 3d = \lambda \quad \dots \text{ (iii)} \\ \text{and } c = 0 & \dots \text{ (iv); } & 2a + 3b = 0 \quad \dots \text{ (v)} \end{array}$$

Using (ii), (iii), (iv) and (v), we get $a = \lambda, b = \frac{-2\lambda}{3}, c = 0, d = \frac{\lambda}{3}$

Also, $|3A| = 108 \Rightarrow 3^2 |A| = 108 \Rightarrow |A| = 12$

Putting the values of a, b, c and d in (i), we get $\dots$ (vi)

$$A = \begin{bmatrix} \lambda & \frac{-2\lambda}{3} \\ 0 & \frac{\lambda}{3} \end{bmatrix} \Rightarrow |A| = \frac{\lambda^2}{3} \quad \dots \text{ (vii)}$$

From (vi) and (vii), we have $\frac{\lambda^2}{3} = 12 \Rightarrow \lambda^2 = 36 \Rightarrow \lambda = \pm 6$

Put $\lambda = 6$, we get $A = \begin{bmatrix} 6 & -4 \\ 0 & 2 \end{bmatrix} \therefore A^2 = \begin{bmatrix} 6 & -4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 6 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 36 & -24-8 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 36 & -32 \\ 0 & 4 \end{bmatrix}$

4.(D) For unique solution $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$

$$\Rightarrow 1(k+2) - 1(2k+3) + 1(4-3) \neq 0 \Rightarrow -k + 2 - 3 + 1 \neq 0 \Rightarrow k \neq 0 \therefore S = R - \{0\}$$

5.(D) The given system of equations has no solution $\therefore \begin{vmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{vmatrix} = 0$

Applying $R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$

$$\Rightarrow \begin{vmatrix} 0 & a-2 & -1 \\ 0 & -3 & -1 \\ 1 & 5 & 3 \end{vmatrix} = 0 \Rightarrow a = -1$$

Now, for no solution, $(adj A)B \neq O$

$$\therefore adj(A) = \begin{bmatrix} -4 & 8 & -4 \\ -1 & 2 & -1 \\ 3 & -6 & 3 \end{bmatrix}$$

$$\Rightarrow (adj A)(B) = \begin{bmatrix} -4 & 8 & -4 \\ -1 & 2 & -1 \\ 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix} \neq O \Rightarrow -12 + 48 - 4b \neq 0 \Rightarrow b \neq 9$$

6.(D) Given, $(A - 3I)(A - 5I) = O \therefore A^2 - 8A + 15I = O$

Post multiplying by A^{-1} on both sides, we have

$$A \cdot AA^{-1} - 8A \cdot A^{-1} + 15I \cdot A^{-1} = O$$

$$\Rightarrow A - 8I + 15A^{-1} = O \Rightarrow A + 15A^{-1} = 8I \Rightarrow \frac{1}{2}A + \frac{15}{2}A^{-1} = 4I \quad \dots \text{ (i)}$$

Comparing (i) with $\alpha A + \beta A^{-1} = 4I$, we get $\alpha = \frac{1}{2}$ and $\beta = \frac{15}{2}$

$$\therefore \alpha + \beta = \frac{1}{2} + \frac{15}{2} = 8$$

$$7.(C) \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}; A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}; A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 4 & 1 \end{bmatrix}; A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 15 & 5 & 1 \end{bmatrix}, \dots, A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 20 & 1 & 0 \\ 210 & 20 & 1 \end{bmatrix}$$

$$\text{Given, } B = A^{20} \left\{ \therefore a_{31} \text{ in } A^n = \sum_{i=1}^3 a_{i1} \text{ of } A^{n-1} \text{ and } a_{21} \text{ in } A^n = a_{32} \text{ in } A^n = n \right\}$$

$$\therefore \text{Sum of the elements of the first column of } B = 1 + 20 + 210 = 231$$

$$8.(A) \quad \text{Given, } A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

Then, A satisfies the characteristic equation $A^2 - 3A - 10I = 0$

$$\text{Now } 3A^2 + 12A = 3(3A + 10I) + 12A = 21A + 30I$$

$$= \begin{bmatrix} 42 & -63 \\ -84 & 21 \end{bmatrix} + \begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$

$$\therefore \text{adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$$

$$9.(C) \quad \text{The equation can be written as } \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a-1 & 0 \\ a & b & 1 \end{vmatrix} = -(1-a)^2$$

$$\text{The necessary condition is } \Delta = 0 \Rightarrow a = 1$$

But for $a = 1$ the equation becomes

$$x + y + z = 1$$

$$x + y + z = 1$$

$$x + by + z = 0$$

For no solution $b = 1$. Then S is a singleton set.

$$10.(D) \quad \text{Let } A = \begin{bmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{bmatrix}$$

$$\therefore |A| = 0(0 - \cos x \sin x) - \cos x(0 - \cos^2 x) - \sin x(\sin^2 x - 0) = 0$$

$$\Rightarrow \cos^3 x - \sin^3 x = 0$$

$$\Rightarrow \tan^3 x = 1 \Rightarrow \tan x = 1$$

$$\sum_{x \in S} \frac{\sqrt{3} + \tan x}{1 - \sqrt{3} \tan x} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}} \times \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \quad (\because \tan x = 1)$$

$$\Rightarrow \frac{1 + 3 + 2\sqrt{3}}{-2} = \frac{4}{-2} - \frac{2\sqrt{3}}{2} = -2 - \sqrt{3}$$

11.(B) Let $A = \begin{bmatrix} 2 & 4 & -\lambda \\ 4 & \lambda & 2 \\ \lambda & 2 & 2 \end{bmatrix}$

For the system of linear equations to have infinitely many solutions, $|A| = 0$

$$\Rightarrow \begin{vmatrix} 2 & 4 & \lambda \\ 4 & \lambda & 2 \\ \lambda & 2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 2(2\lambda - 4) - 4(8 - 2\lambda) - \lambda(8 - \lambda^2) = 0$$

$$\Rightarrow \lambda^3 + 4\lambda - 40 = 0$$

$\therefore$ Number of solutions = 1 (lies between 3 and 4)

12.(A)

13.(A) $A + B = 2B' \Rightarrow (A + B)' = (2B')' \Rightarrow A' + B' = 2B$

$$\Rightarrow B = \frac{A' + B'}{2}$$

Now, $A + \left(\frac{B' + A'}{2}\right) = 2B' \quad [\because A + B = 2B']$

$$\Rightarrow 2A + B' + A' = 4B' \Rightarrow 2A + A' = 3B' \Rightarrow A = \frac{3B' - A'}{2}$$

Also, $3A + 2B = I_3 \quad \dots (1)$

$$\Rightarrow 3\left(\frac{3B' - A'}{2}\right) + 2\left(\frac{A' + B'}{2}\right) = I_3$$

$$\Rightarrow \left(\frac{9B' + 2B'}{2}\right) + \left(\frac{2A' - 3A'}{2}\right) = I_3$$

$$\Rightarrow 11B' - A' = 2I_3 \Rightarrow (11B' - A')' = (2I_3)'$$

$$\Rightarrow 11B - A = 2I_3 \quad \dots (2)$$

Multiplying (2) by 3 and then adding (1) and (2), we get

$$35B = 7I_3 \Rightarrow B = \frac{I_3}{5}$$

From (2), $11\frac{I_3}{5} - A = 2I_3 \Rightarrow 11\frac{I_3}{5} - 2I_3 = A \Rightarrow A = \frac{I_3}{5}$

$$\therefore 5A = 5B = I_3 \Rightarrow 10A + 5B = 3I_3$$

14.(B) We have $AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix} = \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$

$A(\text{adj } A) = AA^T$ is known, so equating the two expressions,

$$\begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix} = \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix}$$

We have, $10a + 3b = 13$ and $15a - 2b = 0$

On solving, we get $a = 2/5$, $b = 3$

Then, $5a + b = 2 + 3 = 5$

15.(D) The system $AX = 0$ has non-trivial solution iff $\det A = 0$ i.e., $\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow (\lambda + 1) - \lambda(-\lambda^2 + 1) - (\lambda + 1) = 0$$

$$\Rightarrow \lambda^3 - \lambda = 0 \Rightarrow \lambda(\lambda^2 - 1) = 0 \therefore \lambda = 0, 1, -1$$

16.(C) We have, $\begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$

Applying $R_1 \rightarrow R_1 - R_3, R_2 \rightarrow R_2 - R_3$ $\begin{vmatrix} \cos x - \sin x & 0 & \sin x - \cos x \\ 0 & \cos x - \sin x & \sin x - \cos x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$

Applying $C_3 \rightarrow C_3 + C_2$ $\begin{vmatrix} \cos x - \sin x & 0 & \sin x - \cos x \\ 0 & \cos x - \sin x & 0 \\ \sin x & \sin x & \sin x + \cos x \end{vmatrix} = 0$

Expanding along first column, we get $(\sin x - \cos x)^2(2 \sin x + \cos x) = 0$

$$\Rightarrow \cos x = -2 \sin x \text{ or } \cos x = \sin x$$

$$\Rightarrow \tan x = \frac{-1}{2} \text{ or } \tan x = 1 \Rightarrow x = -\tan^{-1}\left(\frac{1}{2}\right), \frac{\pi}{4} \therefore \text{Two solutions.}$$

17.(C) $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, P^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$$PP^T = P^TP = I$$

$$Q^{2015} = (PAP^T)(PAP^T) \dots (2015 \text{ terms}) = PA^{2015}P^T P^T Q^{2015}P = A^{2015}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^{2015} = \begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix} \text{ So, } P^T Q^{2015} P = \begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$$

18.(A) We have, $A^2 - 5A = -7I$

$$\Rightarrow AAA^{-1} - 5AA^{-1} = -7IA^{-1} \Rightarrow AI - 5I = -7A^{-1} \Rightarrow A^{-1} = \frac{1}{7}(5I - A)$$

Also, $A^3 - 2A^2 - 3A + I = A(5A - 7I) - 2A^2 - 3A + I$
 $= 5A^2 - 7A - 2A^2 - 3A + I = 3A^2 - 10A + I = 3(5A - 7I) - 10A + I = 5A - 20I = 5(A - 4I)$

19.(D) We have, $A = \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ -9 & -2 \end{bmatrix}$

Also, $|A| = -1$

Now, $A^{2016} - 2A^{2015} - A^{2014} = A^{2014}(A^2 - 2A - I)$

$$\therefore |A^{2016} - 2A^{2015} - A^{2014}| = |A^{2014}| |A^2 - 2A - I| = |A|^{2014} \begin{vmatrix} 20 & 5 \\ -15 & -5 \end{vmatrix} = -25$$

20.(A) The system is $(2 - \lambda)x_1 - 2x_2 + x_3 = 0$

$$2x_1 - (3 + \lambda)x_2 + 2x_3 = 0 \quad -x_1 + 2x_2 - \lambda x_3 = 0$$

For non-trivial solution, the determinant of the coefficient matrix must vanish. Then

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2 - \lambda)\{\lambda(3 + \lambda) - 4\} + 2\{-2\lambda - 3\} + 1\{4 - (3 + \lambda)\} = 0$$

$$\Rightarrow (2 - \lambda)(\lambda^2 + 3\lambda - 4) - 4\lambda - 6 + 1 - \lambda = 0 \quad \Rightarrow (2 - \lambda)(\lambda^2 + 3\lambda - 4) - 5\lambda - 5 = 0$$

$$\Rightarrow (2 - \lambda)(\lambda - 1)(\lambda + 4) - 5(\lambda + 1) = 0 \quad \Rightarrow (\lambda - 1)(-\lambda^2 - 2\lambda + 8 - 5) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 + 2\lambda - 3) = 0 \quad \Rightarrow (\lambda - 1)^2(\lambda + 3) = 0$$

Thus, $\lambda = 1, 1, -3$ $\therefore$ Set of all λ 's contain 2 elements.

21.(B) As $AA^T = 9I$, we have $\left(\frac{A}{3}\right)\left(\frac{A}{3}\right)^T = I$

$$\text{Hence, } \frac{1}{3}A \text{ is an orthogonal matrix. Now } \frac{1}{3}A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{a}{3} & \frac{2}{3} & \frac{b}{3} \end{bmatrix}$$

We know that row (column) form mutually orthogonal unit vectors.

Then $\left(\frac{a}{3}, \frac{2}{3}, \frac{b}{3}\right)$ is a unit vector, gives $a^2 + 4 + b^2 = 9$

Also, $a + 2b + 4 = 0$ and $2a - 2b + 2 = 0$

The solution is $(-2, -1)$, which is consistent with all the equations.

22.(C) $\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix} = xyz - (x + y + z) + 2$

Since $A.M. \geq G.M.$

$$\Rightarrow \frac{x+y+z}{3} \geq (xyz)^{1/3} \quad \Rightarrow \quad x + y + z \geq 3(xyz)^{1/3}$$

For least value of xyz , $xyz - 3(xyz)^{1/3} + 2 \geq 0$

$$\Rightarrow t^3 - 3t + 2 \geq 0 \quad (\text{Put } t = (xyz)^{1/3}) \quad \Rightarrow \quad (t + 2)(t^2 - 2t + 1) \geq 0 \quad \Rightarrow \quad t = -2, 1$$

So, least value of $t^3 = xyz$ is -8

23.(C) Given that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\text{So, } A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A^2 = -I$$

$$A^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

If we check, then options **(A)**, **(B)**, and **(D)** are correct.

$$\text{Now, for option (C), } A^2 + I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A(A^2 - I) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

24.(A) Given that $|5 \text{adj } A| = 5$

$$\begin{aligned} \Rightarrow 5^3 | \text{adj } A | &= 5 \Rightarrow | \text{adj } A | = \frac{1}{5^2} \\ \Rightarrow | A |^{3-1} &= \frac{1}{5^2} \quad (\because | \text{adj } A | = | A |^{n-1}) \\ \Rightarrow | A | &= \pm \frac{1}{5} \end{aligned}$$

25.(B) Put $x = 1$ on both sides, we get $\begin{vmatrix} 2 & 2 & -1 \\ 4 & 3 & 0 \\ 6 & 1 & 1 \end{vmatrix} = a - 12 \Rightarrow a = 24$

26.(A) $BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$
 $= (A^{-1}A^T)((A^T)^T(A^{-1})^T) = (A^{-1}A^T)(A(A^{-1})^T)$
 $= A^{-1}(A^T A)(A^T)^{-1} = A^{-1}(AA^T)(A^T)^{-1} = (A^{-1}A)(A^T)(A^T)^{-1} = I \cdot I = I$

Recall that $(AB)^T = B^T A^T$ and that the matrix multiplication is associative.

27.(B) $\begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$ (By multiplication of determinants)

$$= [(1-\alpha)(1-\beta)(\beta-\alpha)]^2$$

On comparison, $K = 1$

28.(B) If A is non-singular matrix, then $|A| \neq 0$.

$$AA^T = A^T \text{ and } B = A^{-1}A^T$$

$$\begin{aligned} BB^T &= (A^{-1}A^T)(A^{-1}A^T)^T = A^{-1}A^T A(A^{-1})^T & [\because (A')^T &= A] \\ &= A^{-1}AA^T(A^{-1})^T & [\because AA^T &= A^T A] \\ &= IA^T(A^{-1})^T = A^T(A^{-1})^T = (A^{-1}A)^T & [\because (AB)^T &= B^T A^T] \\ &= I^T = I \end{aligned}$$

29.(B) Given equations can be written in matrix form $AX = B$

Where, $A = \begin{bmatrix} k+1 & 8 \\ k & k+3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix}$

For no solution, $|A| = 0$ and $(\text{adj } A)B \neq 0$

Now, $|A| = \begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix} = 0$

$$\Rightarrow (k+1)(k+3) - 8k = 0 \Rightarrow k^2 + 4k + 3 - 8k = 0 \Rightarrow k^2 - 4k + 3 = 0$$

$$\Rightarrow (k-1)(k-3) = 0 \Rightarrow k = 1 \text{ or } 3$$

Now, $\text{adj } A = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix}$

Now, $(\text{adj } A)B = \begin{bmatrix} k+3 & -8 \\ -k & k+1 \end{bmatrix} \begin{bmatrix} 4k \\ 3k-1 \end{bmatrix} = \begin{bmatrix} (k+3)(4k) - 8(3k-1) \\ -4k^2 + (k+1)(3k-1) \end{bmatrix} = \begin{bmatrix} 4k^2 - 12k + 8 \\ -k^2 + 2k - 1 \end{bmatrix}$

$$\text{Put } k = 1: (\text{adj } A)B = \begin{bmatrix} 4-12+8 \\ -1+2-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{not true})$$

$$\text{Put } k = 3: (\text{adj } A)B = \begin{bmatrix} 36-36+8 \\ -9+6-1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix} \neq 0 \quad (\text{true})$$

Hence, required value of k is 1.

Alternate Solution : Condition for the system of equations has no solution.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \quad \therefore \quad \frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$$

$$\text{Take } \frac{k+1}{k} = \frac{8}{k+3} \Rightarrow k^2 + 4k + 3 = 8k \Rightarrow k^2 - 4k + 3 = 0 \Rightarrow (k-1)(k-3) = 0$$

$$k = 1, 3$$

$$\text{If } k = 1, \text{ then } \frac{8}{1+3} \neq \frac{4 \cdot 1}{2} \quad (\text{not true}) \quad \text{If } k = 3, \text{ then } \frac{8}{6} \neq \frac{4 \cdot 3}{9-1} \quad (\text{true})$$

Hence, only one value of k exists.

30.(B) If A is matrix of order n , then $|\text{adj } A| = |A|^{n-1}$

$$\text{Given, } P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix} \quad \therefore \quad |P| = 1(12-12) - \alpha(4-6) + 3(4-6) = 2\alpha - 6$$

$$\therefore \quad P = \text{adj}(A) \quad [\text{Given}]$$

$$\therefore \quad |P| = |\text{adj } A| = |A|^2 = 16 \Rightarrow 2\alpha - 6 = 16 \Rightarrow 2\alpha = 22 \quad \therefore \quad \alpha = 11$$

$$\text{31.(D) Given matrices are } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

To find matrix $u_1 + u_2$. Since both Au_1 and Au_2 are given, hence adding them we get :

$$A(u_1 + u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 0+1 \\ 0+0 \end{bmatrix} \Rightarrow A(u_1 + u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Since, A is a non-singular matrix, i.e., $|A| \neq 0$, then, we get :

$$A^{-1}A(u_1 + u_2) = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \dots\dots (i)$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 0 + 0 \quad [\text{by expanding the determinant along row 1}]$$

$$\Rightarrow |A| = 1$$

Now, cofactor matrix of A (i.e., the matrix in which every element is replaced by corresponding cofactor)

$$= \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \quad \text{adj}(A) = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{\text{adj}(A)}{|A|} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad [\because |A| = 1]$$

From Equation (i), we get : $u_1 + u_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \therefore u_1 + u_2 = \begin{bmatrix} 1+0+0 \\ -2+1+0 \\ 1-2+0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

32.(C) Given :

(i) Two matrices P and Q of order 3×3 such that $P \neq Q$

(ii) $P^3 = Q^3$ and $P^2Q = Q^2P$

To find the value of determinant of $P^2 + Q^2$. On subtracting the given equations, we get : $P^3 - P^2Q = Q^3 - Q^2P$

$$\Rightarrow P^2(P - Q) = Q^2(Q - P) \Rightarrow (P - Q)(P^2 + Q^2) = 0$$

Now, since $P \neq Q$ [Given]

$$\Rightarrow P - Q \neq 0 \Rightarrow |P^2 + Q^2| = 0 \Rightarrow P^2 + Q^2 = 0$$

33.(A) Since, equation has non-zero solution.

$$\Rightarrow \Delta = 0 \quad \therefore \quad \begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4(4 - 2) - k(k - 2) + 2(2k - 8) = 0 \Rightarrow 8 - k^2 + 2k + 4k - 16 = 0 \Rightarrow k^2 - 6k + 8 = 0$$

$$\Rightarrow (k - 2)(k - 4) = 0 \Rightarrow k = 2, 4$$

Hence, number of values of k is two.

34.(A) Since A and B are symmetric matrices $\therefore A^T = A$ and $B^T = B$

Now, to check $A(BA)$ is symmetric.

$$\text{Consider } [A(BA)]^T = (BA)^T \cdot A^T = (A^T B^T) A^T = (AB)A = A(BA)$$

$$\text{So, } [A(BA)]^T = A(BA) \Rightarrow A(BA) \text{ is symmetric}$$

Similarly, $(AB)A$ is symmetric.

So, Statement I is true. Also, Statement II is true, as if A and B are symmetric.

$$\Rightarrow (AB) \text{ is symmetric, if } AB = BA \quad \text{i.e., } AB \text{ commutative}$$

Hence, both the Statements are true but Statement II is not a correct explanation of Statement I.

35.(B) $x - ky + z = 0$, $kx + 3y - kz = 0$ and $3x + y - z = 0$ has trivial solution.

$$\Rightarrow \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} \neq 0 \Rightarrow 1(-3+k) + k(-k+3k) + 1(k-9) \neq 0 \Rightarrow k-3+2k^2+k-9 \neq 0$$

$$\Rightarrow 2k^2+2k-12 \neq 0 \Rightarrow k^2+k-6 \neq 0 \Rightarrow (k+3)(k-2) \neq 0$$

$$\therefore k \neq 2, -3$$

$$k \in R - \{2, -3\}$$

36.(A) Determinant of skew-symmetric matrix of odd order is zero and of even order is perfect square.
So, Statement I is true.

Now, $\det(A^T) = \det(A)$ and $\det(-A) = (-1)^n \det(A)$ So, Statement II is false.

37.(A) Here, $H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$

$$\therefore H^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \Rightarrow H^K = \begin{bmatrix} \omega^K & 0 \\ 0 & \omega^K \end{bmatrix}$$

Then, $H^{K+1} = \begin{bmatrix} \omega^{K+1} & 0 \\ 0 & \omega^{K+1} \end{bmatrix} \therefore H^{70} = \begin{bmatrix} \omega^{70} & 0 \\ 0 & \omega^{70} \end{bmatrix} = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = H$

38.(D) The given system of linear equations can be put in the matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -8 \end{bmatrix} \quad \left[\begin{array}{l} \text{by } R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \right]$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \quad [\text{by } R_3 \rightarrow R_3 - R_2]$$

Clearly, the given system of equations from solution.

Alternate Solution :

Subtracting the addition of first two equations has no third equation, we get : $0 = -5$, which is an absurd result.

Hence, the given system of equations has no solution.

39.(D) Consider $\begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix}$. By placing 1 in anyone of the 6* position and 0 elsewhere, we get 6 non-singular matrices.

Similarly, $\begin{pmatrix} * & * & 1 \\ * & 1 & * \\ 1 & * & * \end{pmatrix}$ gives atleast one non-singular matrix. Hence, we get atleast 7 non-singular matrix.

40.(D) A satisfies $A^2 - \text{tr}(A) \cdot A + (\det A)I = 0$

On comparing with $A^2 - I = 0$, we get : $\text{tr}(A) = 0$, $|A| = -1$

Therefore, Statement I is true but Statement II is false.

Alternative Solution :

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \neq 0$

Now, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$\therefore A^2 = I \Rightarrow a^2+bc=1, bc+d^2=1$ and $ab+bd=ac+cd=0$

Also, $c \neq 0$ and $b \neq 0 \Rightarrow a+d=0, \text{tr}(A) = a+d=0$ and $|A| = ad-bc = -a^2-bc = -1$

Hence, Statement I is true but Statement II is false.

41.(C) $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix} = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix} \quad \left[\because |A| = |A^T| \right]$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix} \quad [C_2 \leftrightarrow C_3]$

$= [1 + (-1)^{n+2}] \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$. This is equal to zero only, if $n+2$ is odd i.e., n is an odd integer.

42.(C) $|adj A| = |A|^{n-1} = |A|^{2-1} = |A|$

$adj(adj A) = |A|^{n-2} A = |A|^0 A = A$

Hence, both the statements are true but Statement II is not a correct explanation of Statement I.

43.(D) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [\because A^2 = I]$

$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow b(a+d)=0, c(a+d)=0$ and $a^2+bc=1$

$bc+d^2=1 \Rightarrow a=1$

$d=-1$

$b=c=0$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, A \neq I, A \neq -I$

$\det(A) = -1$ (Statement I is true). Statement II, $\text{tr}(A) = 1-1=0$, Statement II is false.

44.(A) Given equation are $x-cy-bz=0, cx-y+az=0$ and $bx+ay-z=0$

For non-zero solution, $\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0 \Rightarrow 1(1-a^2) + c(-c-ab) - b(ac+b) = 0 \Rightarrow 1-a^2-c^2-abc-abc-b^2=0$

$\Rightarrow a^2+b^2+c^2+2abc=1$

45.(D) As $\det(A) = \pm 1, A^{-1}$ exists

And $A^{-1} = \frac{1}{\det(A)} (adj A) = \pm (adj A)$

All entries in $\text{adj}(A)$ are integers. Hence, A^{-1} has integer entries.

46.(B) Given that, $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = 0$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$,

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} = xy$$

Hence, D is Divisible by both x and y .

47.(C) Since, $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \quad \therefore \quad A^2 = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 25\alpha + 5\alpha^2 & 10\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 5\alpha^2 + 25\alpha \\ 0 & 0 & 25 \end{bmatrix}$

$$\Rightarrow |A^2| = \begin{vmatrix} 25 & 25\alpha + 5\alpha^2 & 10\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 5\alpha^2 + 25\alpha \\ 0 & 0 & 25 \end{vmatrix} = 25 \begin{vmatrix} \alpha^2 & 25\alpha + 5\alpha^2 \\ 0 & 25 \end{vmatrix} = 625\alpha^2$$

$$\Rightarrow 625\alpha^2 = 25 \quad [\text{Given}] \quad \therefore |\alpha| = \frac{1}{5}$$

48.(A) Since, $A^2 - B^2 = (A - B)(A + B) \quad \therefore \quad A^2 - B^2 = A^2 - B^2 + AB - BA \quad \Rightarrow \quad AB = BA$

49.(C) Given that, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Now, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$ and $BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$

If $AB = BA$, then $a = b$. Hence, $AB = BA$ is possible for infinitely many values of B 's.

50.(A) $\therefore A^2 - A + I = O \Rightarrow A^{-1}A^2 - A^{-1}A + A^{-1}I = O \Rightarrow (A^{-1}A)A - (A^{-1}A) + A^{-1} = O$
 $\Rightarrow A - I + A^{-1} = O \Rightarrow A^{-1} = I - A$

51.(A) Given that, $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get : $f(x) = \begin{vmatrix} 1+a^2x+x+b^2x+x+c^2x & (1+b^2)x & (1+c^2)x \\ x+a^2x+1+b^2x+x+c^2x & 1+b^2x & (1+c^2)x \\ x+a^2x+x+b^2x+1+c^2x & (1+b^2)x & 1+c^2x \end{vmatrix}$

$$= \begin{vmatrix} 1+(a^2+b^2+c^2+2)x & (1+b^2)x & (1+c^2)x \\ 1+(a^2+b^2+c^2+2)x & 1+b^2x & (1+c^2)x \\ 1+(a^2+b^2+c^2+2)x & (1+b^2)x & 1+c^2x \end{vmatrix} = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} \quad [\because a^2+b^2+c^2 = -2, \text{ given}]$$

Applying $R_1 \rightarrow R_1 - R_3, R_2 \rightarrow R_2 - R_3$, we get :
$$\begin{bmatrix} 0 & 0 & x-1 \\ 0 & 1-x & x-1 \\ 1 & (1+b^2)x & 1+c^2x \end{bmatrix} = \begin{bmatrix} 0 & x-1 \\ 1-x & x-1 \end{bmatrix} = (x-1)^2$$

Hence, $f(x)$ is of degree 2.

52.(D) The system of given equations has no solution, if
$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$,
$$\begin{vmatrix} \alpha+2 & 1 & 1 \\ \alpha+2 & \alpha & 1 \\ \alpha+2 & 1 & \alpha \end{vmatrix} = 0$$

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1, (\alpha+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & \alpha-1 \end{vmatrix} = 0$

$\Rightarrow (\alpha+2)(\alpha-1)^2 = 0 \quad \therefore \alpha = 1, -2$

53.(C) Since, $a_1, a_2, \dots, a_n, \dots$ are in GP, then $\log a_n, \log a_{n+1}, \log a_{n+2}, \dots, \log a_{n+8}, \dots$ are in AP

Given that,
$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix} \quad \therefore \Delta = \begin{vmatrix} a & a+d & a+2d \\ a+3d & a+4d & a+5d \\ a+6d & a+7d & a+8d \end{vmatrix} = 0$$

Where a and d are the first term and common difference of an AP.

Applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1 \Rightarrow \Delta = \begin{vmatrix} a & d & 2d \\ a+3d & d & 2d \\ a+6d & d & 2d \end{vmatrix} = 0$ [Since, two columns are similar]

54.(D) The given matrix, $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

(A) It is clear that A is not zero matrix

(B) Now, $(-1) = -1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \neq A \quad \therefore (-1) \neq A$

(C) Now, $|A| = 0 + 0 - 1 \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1(-1) = 1$. Since, $|A| \neq 0$, so A^{-1} exists.

(D) Now, $A^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore A^2 = I$

55.(D) Since, B is the inverse of matrix A , i.e., $B = A^{-1}$.

$\therefore 10A^{-1} = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$

$$\Rightarrow 10A^{-1}A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} A \Rightarrow 10I = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \quad [\because A^{-1}A = I]$$

$$\Rightarrow \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ -5+\alpha & 5+\alpha & -5+\alpha \\ 0 & 0 & 10 \end{bmatrix} \Rightarrow -5+\alpha = 0 \text{ and } 5+\alpha = 10 \Rightarrow \alpha = 5$$

56.(C) Since, the system of linear equation has a non-zero solution, then $\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$, we get :

$$\Rightarrow \begin{vmatrix} 1 & 2a & a \\ 0 & 3b-2a & b-a \\ 0 & 4c-2a & c-a \end{vmatrix} = 0 \Rightarrow (3b-2a)(c-a) - (4c-2a)(b-a) = 0$$

$$\Rightarrow 3bc - 3ba - 2ac + 2a^2 = 4bc - 2ab - 4ac + 2a^2 \Rightarrow 2ac = bc + ab$$

On dividing by abc both sides, we get $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

So, a , b and c are in HP .

57.(B) Given that, $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$$A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

$$\therefore A^2 = AA = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} a^2+b^2 & ab+ba \\ ba+ab & b^2+a^2 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix} \Rightarrow \alpha = a^2+b^2 \text{ and } \beta = 2ab$$

58.(B) Given that, $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix} = 1(\omega^{3n}-1) - \omega^n(\omega^{2n}-\omega^{2n}) + \omega^{2n}(\omega^n-\omega^{4n})$

$$= 1(1-1) - 0 + \omega^{2n}(\omega^n-\omega^n) \quad [\because \omega^3 = 1]$$

$$= 0$$

59.(D) Since, l , m and n are the p th, q th and r th terms of a GP whose first term is A and common ratio is R .

$$\therefore l = AR^{p-1} \Rightarrow \log l = \log A + (p-1)\log R$$

Similarly, $\log m = \log A + (q-1)\log R$ and $\log n = \log A + (r-1)\log R$

$$\text{Now, } \begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$\text{Applying } C_1 \rightarrow C_1 - [C_3 \log A + (C_2 - C_3)\log R], = \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} = 0$$

60.(A) Let $\Delta = \begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega-i & -1 \end{vmatrix}$

Applying $R_1 \rightarrow R_1 + R_3$, we get : $\begin{vmatrix} 1-i & -1 & \omega^2-1 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega-i & -1 \end{vmatrix} \quad \left[\because \omega + \omega^2 = -1 \right]$

$= 0$ [Since, two rows are identical]

61.(B) Given matrix A is a symmetric and matrix B is a skew-symmetric.

$\therefore A^T = A$ and $B^T = -B$

Since, $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$ (given) ... (i)

On taking transpose both sides, we get

$(A + B)^T = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}^T$

$\Rightarrow A^T + B^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$... (ii)

Given, $A^T = A$ and $B^T = -B$

$\Rightarrow A - B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

On solving equation (i) and (ii), we get

$A = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

So, $AB = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

62.(B) Given matrix

$A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}, (x, y \in R, x \neq y)$

For which

$A^T A = 3I_3$

$\Rightarrow \begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 8x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Here, two matrices are equal, therefore equating the corresponding elements, we get

$8x^2 = 3$ and $6y^2 = 3$

$$\Rightarrow x = \pm\sqrt{\frac{3}{8}} \text{ and } y = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ There are 2 different values of x and y each.

So, 4 matrices are possible such that $A^T A = 3I_3$

63.(C) Given, matrix $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

$$\begin{aligned} \therefore A^2 &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \alpha - \sin^2 \alpha & -\cos \alpha \sin \alpha - \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha + \cos \alpha \sin \alpha & -\sin^2 \alpha + \cos^2 \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix} \end{aligned}$$

Similarly,

$$A^n = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}, n \in \mathbb{N}$$

$$\Rightarrow A^{32} = \begin{bmatrix} \cos(32\alpha) & -\sin(32\alpha) \\ \sin(32\alpha) & \cos(32\alpha) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (\text{given})$$

So, $\cos(32\alpha) = 0$ and $\sin(32\alpha) = 1$

$$\Rightarrow 32\alpha = \frac{\pi}{2} \quad \Rightarrow \quad \alpha = \frac{\pi}{64}$$

64.(A) Given matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P = X + I \text{ (let)}$$

Now, $P^5 = (I + X)^5$

$$= I + {}^5C_1(X) + {}^5C_2(X^2) + {}^5C_3(X^3) + \dots$$

$$[\because I^n = I, I \cdot A = A \text{ and } (a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x + \dots + {}^nC_n x^n]$$

Here, $X^2 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix}$ and $X^3 = X^2 \cdot X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow X^4 = X^5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $P^5 = I + 5 \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 9 & 3 & 0 \end{bmatrix} + 10 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 135 & 15 & 1 \end{bmatrix}$ and $Q = I + P^5 = \begin{bmatrix} 2 & 0 & 0 \\ 15 & 2 & 0 \\ 135 & 15 & 2 \end{bmatrix} = [q_{ij}]$

$$\Rightarrow q_{21} = 15, q_{31} = 135 \text{ and } q_{32} = 15$$

Here, $\frac{q_{21} + q_{31}}{q_{32}} = \frac{15 + 135}{15} = \frac{150}{15} = 10$

65.(B) Given, $AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix} \begin{bmatrix} 0 & p & p \\ 2q & q & -q \\ r & -r & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0+4q^2+r^2 & 0+2q^2-r^2 & 0-2q^2+r^2 \\ 0+2q^2-r^2 & p^2+q^2+r^2 & p^2-q^2-r^2 \\ 0-2q^2+r^2 & p^2-q^2-r^2 & p^2+q^2+r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that, if two matrices are equal, then corresponding elements are also equal, so

$$4q^2 + r^2 = 1 = p^2 + q^2 + r^2 \quad \dots(i)$$

$$2q^2 - r^2 = 0 \Rightarrow r^2 = 2q^2 \quad \dots(ii)$$

$$\text{and } p^2 - q^2 - r^2 = 0 \quad \dots(iii)$$

Using Equation (ii) and (iii) in equation (i), we get

$$4q^2 + 2q^2 = 1 \quad \dots(iv)$$

$$\Rightarrow 6q^2 = 1$$

$$\Rightarrow 2p^2 = 1 \quad [\text{Using equation (iv)}]$$

$$p^2 = \frac{1}{2} \Rightarrow |p| = \frac{1}{\sqrt{2}}$$

66.(A) Let $\begin{vmatrix} 1 + \cos^2 \theta & \sin^2 \theta & 4 \cos 6\theta \\ \cos^2 \theta & 1 + \sin^2 \theta & 4 \cos 6\theta \\ \cos^2 \theta & \sin^2 \theta & 1 + 4 \cos 6\theta \end{vmatrix} = 0$

Applying $C_1 \rightarrow C_1 + C_2$, we get

$$\Delta = \begin{vmatrix} 2 & \sin^2 \theta & 4 \cos 6\theta \\ 2 & 1 + \sin^2 \theta & 4 \cos 6\theta \\ 1 & \sin^2 \theta & 1 + 4 \cos 6\theta \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_1 - 2R_3$ and $R_2 \rightarrow R_2 - 2R_3$, we get

$$\Delta = \begin{vmatrix} 0 & -\sin^2 \theta & -2 - 4 \cos 6\theta \\ 0 & 1 - \sin^2 \theta & -2 - 4 \cos 6\theta \\ 1 & \sin^2 \theta & 1 + 4 \cos 6\theta \end{vmatrix} = 0$$

On expanding w.r.t. C_1 , we get

$$\Rightarrow \sin^2 \theta (2 + 4 \cos 6\theta) + (2 + 4 \cos 6\theta)(1 - \sin^2 \theta) = 0$$

$$\Rightarrow 2 + 4 \cos 6\theta = 0 \Rightarrow \cos 6\theta = -\frac{1}{2} = \cos \frac{2\pi}{3} \quad \text{or} \quad \cos \frac{4\pi}{3}$$

$$\Rightarrow 6\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow \theta = \frac{\pi}{9}, \frac{2\pi}{9} \quad \left[\because \theta \in \left(0, \frac{\pi}{3}\right) \right]$$

67.(A) Given equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$

On expansion of determinant along R_1 , we get

$$x[(-3x)(x+2) - 2x(x-3)] + 6[2(x+2) + 3(x-3)] - 1[2(2x) - (-3x)(-3)] = 0$$

$$\Rightarrow x[-3x^2 - 6x - 2x^2 + 6x] + 6[2x + 4 + 3x - 9] - 1[4x - 9x] = 0$$

$$\Rightarrow x(-5x^2) + 6(5x - 5) - 1(-5x) = 0 \Rightarrow -5x^3 + 30x - 30 + 5x = 0$$

$$\Rightarrow 5x^3 - 35x + 30 = 0 \Rightarrow x^3 - 7x + 6 = 0$$

Since all roots are real $\therefore$ Sum of roots $= -\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3} = 0$

68.(C) Given determinants are $\Delta_1 = \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$

$$= -x^3 + \sin \theta \cos \theta - \sin \theta \cos \theta + x \cos^2 \theta - x + x \sin^2 \theta = -x^3 \text{ and } \Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}, x \neq 0$$

$$= -x^3 \text{ (similarly as } \Delta_1)$$

So, according to options, we get $\Delta_1 + \Delta_2 = -2x^3$

69.(B) Given $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \dots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3+2+1 \\ 0 & 1 \end{bmatrix},$$

$$\vdots \quad \vdots \quad \vdots$$

$$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \dots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (n-1) + (n-2) + \dots + 3 + 2 + 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{n(n-1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

Since, both matrices are equal, so equating corresponding element, we get

$$\frac{n(n-1)}{2} = 78 \Rightarrow n(n-1) = 156$$

$$= 13 \times 12 = 13(13-1) \Rightarrow n = 13$$

So, $A = \begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix} = A^{-1} = \begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

$$[\because \text{if } |A| = 1 \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}]$$

70.(D) Given, quadratic equation is $x^2 + x + 1 = 0$ having roots α, β .

Then, $\alpha + \beta = -1$ and $\alpha\beta = 1$

Now, given determinant

$$\Delta = \begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} y+1+\alpha+\beta & y+1+\alpha+\beta & y+1+\alpha+\beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

$$\begin{vmatrix} y & y & y \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix} \quad [\because \alpha + \beta = -1]$$

On applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} y & 0 & 0 \\ \alpha & y+\beta-\alpha & 1-\alpha \\ \beta & 1-\beta & y+\alpha-\beta \end{vmatrix}$$

$$= y[(y+(\beta-\alpha))(y-(\beta-\alpha)) - (1-\alpha)(1-\beta)] \quad [\text{Expanding along } R_1]$$

$$= y[y^2 - (\beta-\alpha)^2 - (1-\alpha-\beta+\alpha\beta)]$$

$$= y[y^2 - \beta^2 - \alpha^2 + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$$

$$= y[y^2 - (\alpha+\beta)^2 + 2\alpha\beta + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$$

$$= y[y^2 - 1 + 3 - 1 - 1] = y^3 \quad [\because \alpha + \beta = -1 \text{ and } \alpha\beta = 1]$$

71.(C) Given, matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$, so

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{vmatrix}$$

On applying, $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$,

$$\text{We get } \det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & b-2 & c-2 \\ 4 & b^2-4 & c^2-4 \end{vmatrix}$$

$$= \begin{vmatrix} b-2 & c-2 \\ b^2-4 & c^2-4 \end{vmatrix}$$

$$= \begin{vmatrix} b-2 & c-2 \\ (b-2)(b+2) & (c-2)(c+2) \end{vmatrix}$$

$$= (b-2)(c-2) \begin{vmatrix} 1 & 1 \\ b+2 & c+2 \end{vmatrix}$$

$$[\text{Taking common } (b-2) \text{ from } C_1 \text{ and } (c-2) \text{ from } C_2]$$

$$= (b-2)(c-2)(c-b)$$

Since, 2, b and c are in AP, if assume common difference of AP is d , then

$$b = 2 + d \text{ and } c = 2 + 2d$$

So, $|A| = d(2d)d = 2d^3 \in [2, 16]$ (given)

$$\Rightarrow d^3 \in [1, 8] \Rightarrow d \in [1, 2]$$

$$\therefore 2 + 2d \in [2 + 2, 2 + 4] = [4, 6]$$

$$\Rightarrow c \in [4, 6]$$

72.(A) Given matrix $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$

$$\Rightarrow \det(A) = |A| = \begin{vmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{vmatrix}$$

$$= 1(1 + \sin^2 \theta) - \sin \theta(-\sin \theta + \sin \theta) + 1(\sin^2 \theta + 1)$$

$$\Rightarrow |A| = 2(1 + \sin^2 \theta) \quad \dots(i)$$

As we know that, for $\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$

$$\sin \theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \sin^2 \theta \in \left[0, \frac{1}{2}\right)$$

$$\Rightarrow 1 + \sin^2 \theta \in \left[0 + 1, \frac{1}{2} + 1\right)$$

$$\Rightarrow 1 + \sin^2 \theta \in \left[1, \frac{3}{2}\right)$$

$$\Rightarrow 2(1 + \sin^2 \theta) \in [2, 3)$$

$$\Rightarrow |A| \in [2, 3) \subset \left(\frac{3}{2}, 3\right]$$

73.(B) Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$A = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad \text{(Taking common } (a+b+c) \text{ from } R_1)$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

Now, expanding along R_1 , we get

$$\Delta = (a+b+c) 1 \cdot \{(a+b+c)^2 - 0\} = (a+b+c)^3 = (a+b+c)(a+b+c)^2 \quad \text{(given)}$$

$$\Rightarrow (a+b+c)^2 = (a+b+c)^2$$

$$\Rightarrow a+b+c = \pm(a+b+c)$$

$$\Rightarrow x = -2(a+b+c) \quad [\because x \neq 0]$$

74.(D) Given
$$\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0$$

On applying elementary operations

$C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k - \log_e a_1^r a_2^k & \log_e a_3^r a_4^k - \log_e a_1^r a_2^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k - \log_e a_4^r a_5^k & \log_e a_6^r a_7^k - \log_e a_4^r a_5^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k - \log_e a_7^r a_8^k & \log_e a_9^r a_{10}^k - \log_e a_7^r a_8^k \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \log_e a_1^r a_2^k & \log_e \left(\frac{a_2^r a_3^k}{a_1^r a_2^k} \right) & \log_e \left(\frac{a_3^r a_4^k}{a_1^r a_2^k} \right) \\ \log_e a_4^r a_5^k & \log_e \left(\frac{a_5^r a_6^k}{a_4^r a_5^k} \right) & \log_e \left(\frac{a_6^r a_7^k}{a_4^r a_5^k} \right) \\ \log_e a_7^r a_8^k & \log_e \left(\frac{a_8^r a_9^k}{a_7^r a_8^k} \right) & \log_e \left(\frac{a_9^r a_{10}^k}{a_7^r a_8^k} \right) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \log_e a^{r+k} R^k & \log_e \left(\frac{a^{r+k} R^{r+2k}}{a^{r+k} R^k} \right) & \log_e \left(\frac{a^{r+k} R^{2r+3k}}{a^{r+k} R^k} \right) \\ \log_e a^{r+k} R^{3r+4k} & \log_e \left(\frac{a^{r+k} R^{4r+5k}}{a^{r+k} R^{3r+4k}} \right) & \log_e \left(\frac{a^{r+k} R^{5r+6k}}{a^{r+k} R^{3r+4k}} \right) \\ \log_e a^{r+k} R^{6r+7k} & \log_e \left(\frac{a^{r+k} R^{7r+8k}}{a^{r+k} R^{6r+7k}} \right) & \log_e \left(\frac{a^{r+k} R^{8r+9k}}{a^{r+k} R^{6r+7k}} \right) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \log_e (a^{r+k} R^k) & \log_e R^{r+k} & \log_e R^{2r+2k} \\ \log_e a^{r+k} R^{3r+4k} & \log_e R^{r+k} & \log_e R^{2r+2k} \\ \log_e a^{r+k} R^{6r+7k} & \log_e R^{r+k} & \log_e R^{2r+2k} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \log_e (a^{r+k} R^k) & \log_e R^{r+k} & 2 \log_e R^{r+k} \\ \log_e (a^{r+k} R^{3r+4k}) & \log_e R^{r+k} & 2 \log_e R^{r+k} \\ \log_e (a^{r+k} R^{6r+7k}) & \log_e R^{r+k} & 2 \log_e R^{r+k} \end{vmatrix} = 0$$

$$[\because \log m^n = n \log m \text{ and here } \log_e R^{2r+2k} = \log_e R^{2(r+k)} = 2 \log_e R^{r+k}]$$

$\therefore$ Column C_2 and C_3 are proportional,

So, value of determinant will be zero for any value of $(r, k), r, k \in N$.

$\therefore$ Set 'S' has infinitely many elements.

75.(C) Given matrix, $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2 + 1 & b \\ 1 & b & 2 \end{bmatrix}, b > 0$

$$\begin{aligned} \text{So, } |A| &= \begin{vmatrix} 2 & b & 1 \\ b & b^2 + 1 & b \\ 1 & b & 2 \end{vmatrix} = 2[2(b^2 + 1) - b^2] - b(2b - b) + 1(b^2 - b^2 - 1) \\ &= 2[2b^2 + 2 - b^2] - b^2 - 1 \end{aligned}$$

$$= 2b^2 + 4 - b^2 - 1 = b^2 + 3$$

$$\Rightarrow \frac{\det(A)}{b} = \frac{b^2 + 3}{b} = b + \frac{3}{b}$$

Now, by $AM \geq GM$, we get

$$\frac{b + \frac{3}{b}}{2} \geq \left(b \times \frac{3}{b}\right)^{1/2} \quad \{\because b > 0\}$$

$$\Rightarrow b + \frac{3}{b} \geq 2\sqrt{3}$$

So, minimum value of $\frac{\det(A)}{b} = 2\sqrt{3}$

76.(A) Given, $A = \begin{bmatrix} -2 & 4+d & (\sin \theta) - 2 \\ 1 & (\sin \theta) + 2 & d \\ 5 & (2\sin \theta) - d & (-\sin \theta) + 2 + 2d \end{bmatrix}$

$$\begin{aligned} \therefore |A| &= \begin{vmatrix} -2 & 4+d & (\sin \theta) - 2 \\ 1 & (\sin \theta) + 2 & d \\ 5 & (2\sin \theta) - d & (-\sin \theta) + 2 + 2d \end{vmatrix} \\ &= \begin{vmatrix} -2 & 4+d & (\sin \theta) - 2 \\ 1 & (\sin \theta) + 2 & d \\ 1 & 0 & 0 \end{vmatrix} \quad (R_3 \rightarrow R_3 - 2R_2 + R_1) \\ &= 1[(4+d)d - (\sin \theta + 2)(\sin \theta - 2)] \quad (\text{Expanding along } R_3) \\ &= (d^2 + 4d - \sin^2 \theta + 4) \\ &= (d^2 + 4d + 4) - \sin^2 \theta \\ &= (d+2)^2 - \sin^2 \theta \end{aligned}$$

Note that $|A|$ will be minimum if $\sin^2 \theta$ is maximum i.e., if $\sin^2 \theta$ takes value 1.

$$\therefore |A|_{\min} = 8,$$

Therefore $(d+2)^2 - 1 = 8$

$$\Rightarrow (d+2)^2 = 9$$

$$\Rightarrow d+2 = \pm 3$$

$$\Rightarrow d = 1, -5$$

77.(C) Given matrix B is the inverse matrix of 3×3 matrix A , where $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$

We know that, $\det(A) = \frac{1}{\det(B)}$ $\left[\because \det(A^{-1}) = \frac{1}{\det(A)} \right]$

Since, $\det(A) + 1 = 0$ (given)

$$\frac{1}{\det(B)} + 1 = 0$$

$$\Rightarrow \det(B) = -1$$

$$\Rightarrow 5(-2-3) - 2\alpha(0-\alpha) + 1(0-2\alpha) = -1$$

$$\Rightarrow -25 + 2\alpha^2 - 2\alpha = -1$$

$$\Rightarrow 2\alpha^2 - 2\alpha - 24 = 0$$

$$\Rightarrow \alpha^2 - \alpha - 12 = 0$$

$$\Rightarrow (\alpha - 4)(\alpha + 3) = 0 \quad \Rightarrow \quad \alpha = -3, 4$$

So, required sum of all values of α is $4 - 3 = 1$

$$\begin{aligned} \text{78.(B)} \quad |A| &= \begin{vmatrix} e^t & e^{-t} \cos t & e^{-t} \sin t \\ e^t & -e^{-t} \cos t - e^{-t} \sin t & -e^{-t} \sin t + e^{-t} \cos t \\ e^t & 2e^{-t} \sin t & -2e^{-t} \cos t \end{vmatrix} \\ &= (e^t)(e^{-t})(e^{-t}) \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\cos t - \sin t & -\sin t + \cos t \\ 1 & 2 \sin t & -2 \cos t \end{vmatrix} \quad (\text{taking common from each column}) \end{aligned}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get $[\because e^{t-t} = e^0 = 1]$

$$\begin{aligned} &= e^{-t} \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -2 \cos t - \sin t & -2 \sin t + \cos t \\ 0 & 2 \sin t - \cos t & -2 \cos t - \sin t \end{vmatrix} \\ &= e^{-t} ((2 \cos t + \sin t)^2 + (2 \sin t - \cos t)^2) \quad (\text{expanding along column 1}) \\ &= e^{-t} (5 \cos^2 t + 5 \sin^2 t) \\ &= 5e^{-t} \quad (\because \cos^2 t + \sin^2 t = 1) \end{aligned}$$

$$\Rightarrow |A| = 5e^{-t} \neq 0 \quad \text{for all } t \in R$$

$\therefore A$ is invertible for all $t \in R$ $[\because \text{If } |A| \neq 0, \text{ then } A \text{ is invertible}]$

$$\text{79.(C)} \quad \text{Given, } |ABA^T| = 8$$

$$\Rightarrow |A| |B| |A^T| = 8 \quad [\because |XY| = |X| |Y|]$$

$$\therefore |A|^2 |B| = 8 \quad \dots(i) \quad [\because |A^T| = |A|]$$

$$\text{Also, we have } |AB^{-1}| = 8 \quad \Rightarrow \quad |A| |B^{-1}| = 8$$

$$\Rightarrow \frac{|A|}{|B|} = 8 \quad \dots(ii) \quad \left[\because |A^{-1}| = |A|^{-1} = \frac{1}{|A|} \right]$$

On multiplying equation (i) and (ii), we get

$$|A|^3 = 8 \cdot 8 = 4^3$$

$$\Rightarrow |A| = 4 \quad \Rightarrow \quad |B| = \frac{|A|}{8} = \frac{4}{8} = \frac{1}{2}$$

$$\text{Now, } |BA^{-1}B^T| = |B| \frac{1}{|A|} |B| = \left(\frac{1}{2}\right) \frac{1}{4} \left(\frac{1}{2}\right) = \frac{1}{16}$$

$$\text{80.(C)} \quad \text{We have, } A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\therefore |A| = \cos^2 \theta + \sin^2 \theta = 1$$

$$\text{and } \text{adj } A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$[\because \text{ If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad \left(\because A^{-1} = \frac{\text{adj } A}{|A|} \right)$$

Note that, $A^{-50} = (A^{-1})^{50}$

$$\text{Now, } A^{-2} = (A^{-1})(A^{-1})$$

$$\begin{aligned} \Rightarrow A^{-2} &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & \cos \theta \sin \theta + \sin \theta \cos \theta \\ -\cos \theta \sin \theta - \cos \theta \sin \theta & -\sin^2 \theta + \cos^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \end{aligned}$$

$$\text{Also, } A^{-3} = (A^{-2})(A^{-1})$$

$$\begin{aligned} A^{-3} &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix} \end{aligned}$$

$$\text{Similarly, } A^{-50} = \begin{bmatrix} \cos 50\theta & \sin 50\theta \\ -\sin 50\theta & \cos 50\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos \frac{25}{6}\pi & \sin \frac{25}{6}\pi \\ -\sin \frac{25}{6}\pi & \cos \frac{25}{6}\pi \end{bmatrix} \quad \left(\text{where } \theta = \frac{\pi}{12} \right)$$

$$= \begin{bmatrix} \cos \frac{\pi}{6} & \sin \frac{\pi}{6} \\ -\sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{bmatrix} \quad \left[\because \cos \left(\frac{25\pi}{6} \right) = \cos \left(4\pi + \frac{\pi}{6} \right) = \cos \frac{\pi}{6} \text{ and } \sin \left(\frac{25\pi}{6} \right) = \sin \left(4\pi + \frac{\pi}{6} \right) = \sin \frac{\pi}{6} \right]$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

81.(A) Given system of linear equations is

$$[\sin \theta]x + [-\cos \theta]y = 0 \quad \dots(i)$$

$$\text{and } [\cot \theta]x + y = 0 \quad \dots(ii)$$

Where $[x]$ denotes the greatest integer $\leq x$

$$\text{Here, } \Delta = \begin{vmatrix} [\sin \theta] & [-\cos \theta] \\ [\cot \theta] & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = [\sin \theta] - [-\cos \theta][\cot \theta]$$

$$\text{When } \theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3} \right)$$

$$\sin \theta \in \left(\frac{\sqrt{3}}{2}, 1 \right)$$

$$\Rightarrow [\sin \theta] = 0 \quad \dots(\text{iii})$$

$$-\cos \theta \in \left(0, \frac{1}{2}\right)$$

$$\Rightarrow [-\cos \theta] = 0 \quad \dots(\text{iv})$$

$$\text{and } \cot \theta \in \left(-\frac{1}{\sqrt{3}}, 0\right)$$

$$\Rightarrow [\cot \theta] = -1 \quad \dots(\text{v})$$

$$\text{So, } \Delta = [\sin \theta] - [-\cos \theta][\cot \theta] - (0 \times (-1)) = 0$$

From equations (iii), (iv) and (v)

Thus, for $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$, the given system have infinitely many solutions.

$$\text{When } \theta \in \left(\pi, \frac{7\pi}{6}\right), \sin \theta \in \left(-\frac{1}{2}, 0\right)$$

$$\Rightarrow [\sin \theta] = -1$$

$$-\cos \theta \in \left(\frac{\sqrt{3}}{2}, 1\right) \Rightarrow [\cos \theta] = 0$$

$$\text{and } \cot \theta \in (\sqrt{3}, \infty) \Rightarrow [\cot \theta] = n, n \in \mathbb{N}$$

$$\text{So, } \Delta = -1 - (0 \times n) = -1$$

Thus, for $\theta \in \left(\pi, \frac{7\pi}{6}\right)$, the given system has a unique solution.

82.(C) Given, system of linear equations

$$x + y + z = 6 \quad \dots(\text{i})$$

$$4x + \lambda y - \lambda z = \lambda - 2 \quad \dots(\text{ii})$$

$$\text{and } 3x + 2y - 4z = -5 \quad \dots(\text{iii})$$

has infinitely many solutions, then $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & \lambda & -\lambda \\ 3 & 2 & -4 \end{vmatrix} = 0 \quad \Rightarrow \quad 1(-4\lambda + 2\lambda) - 1(-16 + 3\lambda) + 1(8 - 3\lambda) = 0$$

$$\Rightarrow -8\lambda + 24 = 0$$

$$\Rightarrow \lambda = 3$$

From, the option $\lambda = 3$, satisfy the quadratic equation $\lambda^2 - \lambda - 6 = 0$.

83.(C) Given system of linear equations

$$x + y + z = 5 \quad \dots(\text{i})$$

$$x + 2y + 2z = 6 \quad \dots(\text{ii})$$

$$x + 3y + \lambda z = \mu \quad \dots(\text{iii})$$

The above given system has infinitely many solutions, then the plane represented by these equations intersect each other at a line, means $(x + 3y + \lambda z - \mu)$

$$= p(x + y + z - 5) + q(x + 2y + 2z - 6)$$

$$= (p + q)x + (p + 2q)y + (p + 2q)z - (5p + 6q)$$

On comparing, we get

$$p + q = 1, p + 2q = 3, p + 2q = \lambda$$

$$\begin{aligned} \text{and } 5p + 6q &= \mu \\ \text{So, } (p, q) &= (-1, 2) \\ \Rightarrow \lambda &= 3 \text{ and } \mu = 7 \end{aligned}$$

84.(B) Given system of linear equations

$$2x + 3y - z = 0$$

$$x + ky - 2z = 0$$

and $2x - y + z = 0$ has a non-trivial solution (x, y, z)

$$\begin{aligned} \therefore \Delta = 0 &\Rightarrow \begin{vmatrix} 2 & 3 & -1 \\ 1 & k & -2 \\ 2 & -1 & 1 \end{vmatrix} = 0 \\ &2(k-2) - 3(1+4) - 1(-1-2k) = 0 \\ \Rightarrow 2k - 4 - 15 + 1 + 2k &= 0 \\ \Rightarrow 4k = 18 &\Rightarrow k = \frac{9}{2} \end{aligned}$$

So, system of linear equations is

$$2x + 3y - z = 0 \quad \dots(i)$$

$$2x + 9y - 4z = 0 \quad \dots(ii)$$

$$\text{and } 2x - y + z = 0 \quad \dots(iii)$$

From equations (i) and (ii), we get

$$6y - 3z = 0, \frac{y}{z} = \frac{1}{2}$$

From equation (i) and (iii), we get

$$4x + 2y = 0 \Rightarrow \frac{x}{y} = -\frac{1}{2}$$

$$\text{So, } \frac{x}{z} = \frac{x}{y} \times \frac{y}{z} = -\frac{1}{4} \Rightarrow \frac{z}{x} = -4 \quad \left[\because \frac{y}{z} = \frac{1}{2} \text{ and } \frac{x}{y} = -\frac{1}{2} \right]$$

$$\therefore \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k = -\frac{1}{2} + \frac{1}{2} - 4 + \frac{9}{2} = \frac{1}{2}$$

85.(C) Given system of linear equations

$$x - 2y + kz = 1 \quad \dots(i)$$

$$2x + y + z = 2 \quad \dots(ii)$$

$$\text{and } 3x - y - kz = 3 \quad \dots(iii)$$

has a solution (x, y, z) , $z \neq 0$

On adding equations (i) and (iii), we get

$$x - 2y + kz + 3x - y - kz = 1 + 3$$

$$4x - 3y = 4$$

$$\Rightarrow 4x - 3y - 4 = 0$$

This is the required equations of the straight line in which point (x, y) lies.

86.(B) Given system of linear equations is

$$x - cy - cz = 0,$$

$$cx - y + cz = 0$$

$$\text{and } cx + cy - z = 0$$

We know that a homogeneous system of linear equations have non-trivial solutions iff

$$\Delta = 0$$

$$\begin{aligned}
&\Rightarrow \begin{vmatrix} 1 & -c & -c \\ c & -1 & c \\ c & c & -1 \end{vmatrix} = 0 \\
&\Rightarrow 1(1-c^2) + c(-c-c^2) - c(c^2+c) = 0 \\
&\Rightarrow 1-c^2-c^2-c^3-c^3-c^2 = 0 \\
&\Rightarrow -2c^3-3c^2+1 = 0 \\
&\Rightarrow 2c^3+3c^2-1 = 0 \\
&\Rightarrow (c+1)[2c^2+c-1] = 0 \\
&\Rightarrow (c+1)[2c^2+2c-c-1] = 0 \\
&\Rightarrow (c+1)(2c-1)(c+1) = 0 \\
&\Rightarrow c = -1 \text{ or } \frac{1}{2}
\end{aligned}$$

Clearly, the greatest value of c is $\frac{1}{2}$.

87.(C) The given system of linear equations is

$$x - 2y - z = \lambda x$$

$$x + 2y + z = \lambda y$$

$$-x - y - \lambda z = 0$$

Which can be rewritten as

$$(1-\lambda)x - 2y - z = 0$$

$$\Rightarrow x + (2-\lambda)y + z = 0$$

$$x + y + \lambda z = 0$$

Now, for non-trivial solution, we should have

$$\begin{vmatrix} 1-\lambda & -2 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

[$\because$ If $a_1x + b_1y + c_1z = 0$; $a_2x + b_2y + c_2z = 0$; $a_3x + b_3y + c_3z = 0$ has a non-trivial solution, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)\lambda - 1] + 2[\lambda - 1] - 2[1 - 2 + \lambda] = 0$$

$$\Rightarrow (\lambda - 1)[\lambda^2 - 2\lambda + 1 + 2 - 2] = 0$$

$$\Rightarrow (\lambda - 1)^3 = 0$$

$$\Rightarrow \lambda = 1$$

88.(A) Given system of linear equations,

$$(1 + \alpha)x + \beta y + z = 2$$

$$\alpha x + (1 + \beta)y + z = 3$$

$$\alpha x + \beta y + 2z = 2$$

Has a unique solution, if

$$\begin{vmatrix} 1+\alpha & \beta & 1 \\ \alpha & (1+\beta) & 1 \\ \alpha & \beta & 2 \end{vmatrix} \neq 0$$

Applying $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \alpha & \beta & 2 \end{vmatrix} \neq 0$$

$$\Rightarrow 1(2+\beta) - 0(0+\alpha) - 1(0-\alpha) \neq 0$$

$$\Rightarrow \alpha + \beta + 2 \neq 0 \quad \dots(i)$$

Note that, only (2, 4) satisfy the equations (i.)

89.(A) We know that, if the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

has more than one solution, then $D = 0$ and $D_1 = D_2 = D_3 = 0$. In the given problem.

$$D_1 = 0 \Rightarrow \begin{vmatrix} a & 2 & 3 \\ b & -1 & 5 \\ c & -3 & 2 \end{vmatrix} = 0$$

$$\Rightarrow a(-2+15) - 2(2b-5c) + 3(-3b+c) = 0$$

$$\Rightarrow 13a - 4b + 10c - 9b + 3c = 0$$

$$\Rightarrow 13a - 13b + 13c = 0$$

$$\Rightarrow a - b + c = 0$$

$$\Rightarrow b - a - c = 0$$

90.(A) We know that,

The system of linear equations

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

Has a non-trivial solution, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Now, if the given system of linear equations

$$x + 3y + 7z = 0$$

$$-x + 4y + 7z = 0$$

and $(\sin 3\theta)x + (\cos 2\theta)y + 2z = 0$

has non-trivial solution, then

$$\begin{vmatrix} 1 & 3 & 7 \\ -1 & 4 & 7 \\ \sin 3\theta & \cos 2\theta & 2 \end{vmatrix} = 0$$

$$\Rightarrow 1(8 - 7 \cos 2\theta) - 3(-2 - 7 \sin 3\theta) + 7(-\cos 2\theta - 4 \sin 3\theta) = 0$$

$$\Rightarrow 8 - 7 \cos 2\theta + 6 + 21 \sin 3\theta - 7 \cos 2\theta - 28 \sin 3\theta = 0$$

$$\begin{aligned}
&\Rightarrow -7 \sin 3\theta - 14 \cos 2\theta + 14 = 0 \\
&\Rightarrow -7(3 \sin \theta - 4 \sin^3 \theta) - 14(1 - 2 \sin^2 \theta) + 14 = 0 \\
&[\because \sin 3A = 3 \sin A - 4 \sin^3 A \text{ and } \cos 2A = 1 - 2 \sin^2 A] \\
&\Rightarrow 28 \sin^3 \theta + 28 \sin^2 \theta - 21 \sin \theta - 14 + 14 = 0 \\
&\Rightarrow 7 \sin \theta [4 \sin^2 \theta + 4 \sin \theta - 3] = 0 \\
&\Rightarrow \sin \theta [4 \sin^2 \theta + 6 \sin \theta - 2 \sin \theta - 3] = 0 \\
&\Rightarrow \sin \theta [2 \sin \theta (2 \sin \theta + 3) - 1(2 \sin \theta + 3)] = 0 \\
&\Rightarrow (\sin \theta)(2 \sin \theta - 1)(2 \sin \theta + 3) = 0
\end{aligned}$$

Now, either $\sin \theta = 0$ or $\frac{1}{2}$ $\left[\because \sin \theta \neq -\frac{3}{2} \text{ as } -1 \leq \sin \theta \leq 1 \right]$

In given interval $(0, \pi)$

$$\sin \theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad [\because \sin \theta \neq 0, \theta \in (0, \pi)]$$

Hence, 2 solutions in $(0, \pi)$

91.(A) (a) Here, $D = \begin{vmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{vmatrix} = 1(-27 + 25) + 4(0 - 10) + 7(0 + 6)$ [Expanding along R_1]

$$= -2 - 40 + 42 = 0$$

$\therefore$ The system of linear equations have infinite many solutions.

$[\because \text{system is consistent and does not have unique solution as } D = 0]$

$$\Rightarrow D_1 = D_2 = D_3 = 0$$

Now, $D_1 = 0 \Rightarrow \begin{vmatrix} g & -4 & 7 \\ h & 3 & -5 \\ k & 5 & -9 \end{vmatrix} = 0$

$$\Rightarrow g(-27 + 25) + 4(-9h + 5k) + 7(5h - 3k) = 0$$

$$\Rightarrow -2g - 36h + 20k + 35h - 21k = 0$$

$$\Rightarrow -2g - h - k = 0$$

$$\Rightarrow 2g + h + k = 0$$

92.(D) According to Cramer's rule, here

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & a^2 - 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & a^2 - 3 \end{vmatrix}$$

(Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$)

$$= a^2 - 3 \quad (\text{Expanding along } R_1)$$

$$\text{and } D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 3 & 2 \\ a+1 & 3 & a^2-1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 5 & 3 & -1 \\ a+1 & 3 & a^2-1-3 \end{vmatrix}$$

(Applying $C_3 \rightarrow C_3 - C_2$)

$$= \begin{vmatrix} 2 & 0 & 0 \\ 5 & 3 - \frac{5}{2} & -1 \\ a+1 & 3 - \frac{(a+1)}{2} & a^2 - 1 - 3 \end{vmatrix}$$

(Applying $C_2 \rightarrow C_2 - \frac{1}{2}C_1$)

$$= \begin{vmatrix} 2 & 0 & 0 \\ 5 & 1/2 & -1 \\ a+1 & \frac{5}{2} - \frac{a}{2} & a^2 - 4 \end{vmatrix}$$

$$= 2 \left[\frac{1}{2}(a^2 - 4) + \left(\frac{5}{2} - \frac{a}{2} \right) \right] \quad (\text{Expanding along } R_1)$$

$$= 2 \left[\frac{a^2}{2} - 2 + \frac{5}{2} - \frac{a}{2} \right] = a^2 - 4 + 5 - a = a^2 - a + 1$$

Clearly, when $a = 4$, then $D = 13 \neq 0 \Rightarrow$ unique solution and when $|a| = \sqrt{3}$, then $D = 0$ and $D_1 \neq 0$

$\therefore$ When $|a| = \sqrt{3}$, then the system has no solution i.e. system is inconsistent

93.(A) We have $\begin{vmatrix} 7 & 6 & -2 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{vmatrix} = 7(-20) + 6(20) + 2(10) = 0$

$\therefore$ so infinite solution exists

Now, Equation (i) + 3 eq (iii), we get $10x - 20z = 0 \Rightarrow x = 2z$

94.(D) Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 3^0 a_{11} & 3^1 a_{21} & 3^2 a_{31} \\ 3^1 a_{12} & 3^2 a_{22} & 3^3 a_{32} \\ 3^2 a_{13} & 3^3 a_{23} & 3^4 a_{33} \end{bmatrix}$

$$|B| = 3 \times 3^2 \times 3 \times 3^2 |A| \Rightarrow |A| = \frac{1}{9}$$

95.(13) $x + y + z = 6$

$$x + 2y + 3z = 10$$

$$3x + 2y + \lambda z = \mu$$

In first two equations. Put $z = 0$

$$x + y = 6$$

$$x + 2y = 10, y = 4, x = 2$$

$\therefore (2, 4, 0)$ should satisfy plane (3)

$$6 + 8 = \mu ; \mu = 14$$

Now put $y = 0$

$$x + z = 6 ; x + 3z = 10 ; 2z = 4 \Rightarrow z = 2$$

$\therefore (4, 0, 2)$ is a point on plane (3)

$$12 + 2\lambda = 14 ; 2\lambda = 2 ; \lambda = 1$$

$$\therefore \mu - \lambda^2 = 14 - 1 = 13.$$

96.(D) $A = \begin{bmatrix} 2 & 2 \\ 9 & 4 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A - \lambda I| = 0$ characteristic equation

$$\begin{vmatrix} 2-\lambda & 2 \\ 9 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - 18 = 0; \lambda^2 - 6\lambda + 8 - 18 = 0; \lambda^2 - 6\lambda - 10 = 0; A^2 - 6A - 10I = 0$$

$$A - 6I - 10A^{-1} = 0; 10A^{-1} = A - 6I$$

97.(D) $f(x) = \begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$

Apply $R_1 \Rightarrow R_1 + R_3 - 2R_2$

$$\Rightarrow f(x) = \begin{vmatrix} 1 & 0 & 0 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix} = (x+3)^2 - (x+2)(x+4) = 1 \quad \therefore f(x) = 1 \Rightarrow f(50) = 1$$

98.(B) Given equation,

$$x^2 + x + 1 = 0$$

$$\alpha = \omega, \omega^2$$

Taking $\alpha = \omega$

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}; A^2 = \begin{bmatrix} 1+1+1 & 1+\omega+\omega^2 & 1+\omega^2+\omega \\ 1+\omega+\omega^2 & 1+\omega^2+\omega^4 & 1+\omega^4+\omega^2 \\ 1+\omega^2+\omega & 1+\omega^3+\omega^3 & 1+\omega^4+\omega^3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \Rightarrow A^4 = I$$

Now, $A^{31} = A^{28} \times A^3 = (A^4)^7 \times A^3 \Rightarrow I \times A^3 = A^3$

99.(C) The system of linear equations

$$2x + 2ay + az = 0$$

$$2x + 3by + bz = 0$$

$$2x + 4cy + cz = 0$$

has non zero solution

Then $D = 0$,

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0; \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$$1(3bc - 4bc) - 2a(c - b) + a(4c - 3b) = 0$$

$$\Rightarrow 3bc - 4bc - 2ac + 2ab + 4ac - 3ab = 0 \Rightarrow -bc + 2ac + 2ab - 3ab = 0$$

$$-bc - ab = -2ac$$

$$bc + ab = 2ac$$

$$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in AP}$$

100.(B) $x + 2y + 3z = 1$

$$3x + 4y + 5z = \mu$$

$$4x + 4y + 4z = \delta$$

Here $D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 4 & 4 \end{vmatrix} = 0$

$$D_1 = \begin{vmatrix} 1 & 2 & 3 \\ \mu & 4 & 5 \\ \delta & 4 & 4 \end{vmatrix} = 4\mu - 2\delta - 4 = 2(2\mu - \delta - 2)$$

$$D_2 = \begin{vmatrix} 1 & 1 & 3 \\ 3 & \mu & 5 \\ 4 & \delta & 4 \end{vmatrix} = -8\mu + 4\delta + 8 = -4(2\mu - \delta - 2)$$

$$D_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & \mu \\ 4 & 4 & \delta \end{vmatrix} = 4\mu - 2\delta - 4 = 2(2\mu - \delta - 2)$$

For inconsistent system $D = 0$ and atleast one of D_1, D_2, D_3 should be non zero.

$$\Rightarrow 2\mu - \delta - 2 \neq 0 \quad \text{Now check option}$$

101.(672) Sum of diagonal elements of $AA^T = 3 = a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 + a_{33}^2$

Therefore out of these 9 terms, 6 terms should be zero and remaining 3 terms are either +1 or -1

$$\Rightarrow \text{Number of total possible matrices} = {}^9C_6 \times 2 \times 2 \times 2 = 672$$

102.(A) Given system of equations

$$\lambda x + 2y + 2z = 5; \quad 2\lambda x + 3y + 5z = 8; \quad 4x + \lambda y + 6z = 10$$

$$\begin{vmatrix} \lambda & 2 & 2 \\ 2\lambda & 3 & 5 \\ 4 & \lambda & 6 \end{vmatrix} = 0$$

$$\lambda(18 - 5\lambda) + 2(20 - 12\lambda) + 2(2\lambda^2 - 12) = 0; \quad \lambda^2 + 6\lambda - 16 = 0; \quad \lambda = 2 \text{ and } \lambda = -8$$

$$\text{Now, } D_1 = \begin{vmatrix} 5 & 2 & 2 \\ 8 & 3 & 5 \\ 10 & 2 & 6 \end{vmatrix} \quad D_1 \neq 0$$

Thus for $\lambda = 2$, given set of equations has no solution.

$$\mathbf{103.(B)} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}; \quad |A| = 1(9 + 4) - 1(3 - 4) + 2(-1 - 3) = 13 + 1 - 8 = 6$$

$$B = \text{adj}(A) \quad C = 3A$$

$$\Rightarrow |B| = |\text{adj}(A)| \quad |C| = |3A|$$

$$= |A|^2 = 3^3 |A| = 27 |A|$$

$$\Rightarrow |\text{adj}(B)| = |B|^2 = |A|^4 \quad \therefore \frac{|\text{adj}(B)|}{|C|} = \frac{|A|^4}{27 |A|} = \frac{|A|^3}{27} = \frac{6^3}{27} = 8$$

$$\mathbf{104.(D)} \quad \Delta = 0 \Rightarrow \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & \lambda \\ 1 & \lambda & 1 \end{vmatrix} = 0$$

$$2(-2 - \lambda^2) + 1(1 - \lambda) + 2(\lambda + 2) = 0; \quad -4 - 2\lambda^2 + 1 - \lambda + 2\lambda + 4 = 0$$

$$-2\lambda^2 + \lambda + 1 = 0; \quad 2\lambda^2 - \lambda - 1 = 0; \quad 2\lambda^2 - 2\lambda + \lambda - 1 = 0; \quad \lambda = 1, \lambda = -\frac{1}{2}$$

As can be verified for $\lambda = 1, -\frac{1}{2}$ there's no solutions

105.(A) (P) False

$$\text{Consider} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq I_2$$

$$|A| = 1$$

(Q) True

$$|A| = 1 \Rightarrow \text{Product of diagonal elements} = 1 \Rightarrow \text{True(A)} = 2$$

$$106.(B) |Adj(A)| = 9 = \lambda^2 \Rightarrow |\lambda| = 3$$

$$|B| = 9^2 = 81$$

$$\mu = \frac{1}{|B|} = \frac{1}{81}$$

$$107.(8) 2eq(1) + eq(2) \Rightarrow z = 0$$

$$\text{Put } z \text{ in any eq.} \Rightarrow x - 2y = 0$$

(2y, y, 0) general set

$$15 \leq (2y)^2 + y^2 + 0^2 \leq 150 \Rightarrow 3 \leq y^2 \leq 30$$

$$y^2 = 4, 9, 16, 25$$

$$y = \pm 2, \pm 3, \pm 4, \pm 5$$

$$108.(C) \text{ Given } \Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 2x-3 & 3x-4 & 4x-5 \\ 3x-5 & 5x-8 & 10x-17 \end{vmatrix} = Ax^3 + Bx^2 + Cx + D$$

$$\text{Since } x = 1 \Rightarrow A + B + C + D = \begin{vmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -2 & -3 & -7 \end{vmatrix} = 0$$

$$\text{So, } B + C = -(A + D) \quad \dots(1)$$

$$\begin{aligned} \text{Again } x = 0 \Rightarrow D &= \begin{vmatrix} -2 & -3 & -4 \\ -3 & -4 & -5 \\ -5 & -8 & -17 \end{vmatrix} = - \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 5 & 8 & 17 \end{vmatrix} = -[(136 + 75 + 96) - (80 + 80 + 153)] \\ &= -[307 - 313] = 6 \quad \dots(2) \end{aligned}$$

$$\text{Also, if } x = \frac{1}{t}$$

$$\text{Then } \frac{A}{t^3} + \frac{B}{t^2} + \frac{C}{t} + D = \begin{vmatrix} \frac{1}{t}-2 & \frac{2}{t}-3 & \frac{3}{t}-4 \\ \frac{2}{t}-3 & \frac{3}{t}-4 & \frac{4}{t}-5 \\ \frac{3}{t}-5 & \frac{5}{t}-8 & \frac{10}{t}-17 \end{vmatrix} \Rightarrow A + Bt + Ct^2 + Dt^3 = \begin{vmatrix} 1-2t & 2-3t & 3-4t \\ 2-3t & 3-4t & 4-5t \\ 3-5t & 5-8t & 10-17t \end{vmatrix}$$

$$\text{So, } t = 0 \Rightarrow A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 10 \end{vmatrix} = [(30 + 24 + 30) - (27 + 20 + 40)] = [84 - 87] = -3 \quad \dots(3)$$

$$\text{So, } B + C = -(A + D) = -(-3 + 6) = -3$$

$$109.(10) \quad A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$$

$$a_{11} = x^4 + 3x^2 + 1 = 109 = (x^2 + 1)^2 + x^2 = (x^2 + 1)^2 + (x^2 + 1) - 1$$

$$a_{22} = x^2 + 1 = t(\text{say}) \Rightarrow a_{11} = t^2 + t - 1 = 109$$

$$t^2 + t - 110 = 0$$

$$(t + 11)(t - 10) = 0 \text{ hence } t = -11 \text{ and } 10, t = -11 \text{ is not possible, hence } t = 10$$

$$x^2 + 1 = 10$$

$$110.(A) \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix} = A^2$$

$$\begin{bmatrix} \cos^2 \theta - \sin^2 \theta & 2i \cos \theta \sin \theta \\ 2i \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & i \sin 2\theta \\ i \sin 2\theta & \cos 2\theta \end{bmatrix} = A^2$$

$$A^5 = A^4 \cdot A = \begin{bmatrix} \cos 4\theta & i \sin 4\theta \\ i \sin 4\theta & \cos 4\theta \end{bmatrix} \begin{bmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos 5\theta & i \sin 5\theta \\ i \sin 5\theta & \cos 5\theta \end{bmatrix} \text{ Clearly } a^2 - b^2 = \frac{1}{2} \text{ is wrong}$$

$$111.(5) \Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ 1 & -7 & a \end{vmatrix} = 0 \Rightarrow \boxed{a = 8}$$

$$\Delta_3 = \begin{vmatrix} 1 & -2 & 9 \\ 2 & 1 & b \\ 1 & -7 & 24 \end{vmatrix} = 0 \Rightarrow \boxed{b = 3}$$

We can check $\Delta_1 = 0, \Delta_2 = 0$ for $a = 8, b = 3$; $a - b = 5$

$$112.(C) x + y + z = 2$$

$$2x + 4y - z = 6$$

$$3x + 2y + \lambda z = \mu$$

For infinite solutions

$$D = D_1 = D_2 = D_3 = 0$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 3 & 2 & \lambda \end{vmatrix} = 0$$

$$2\lambda - 1 - 8 = 0 \Rightarrow \lambda = \frac{9}{2}$$

$$D_3 = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 4 & -6 \\ 3 & 2 & -\mu \end{vmatrix} = 0$$

$$2\mu + 6 - 16 = 0 \Rightarrow \mu = 5$$

$$2\lambda + \mu = 14$$

$$113.(D) \text{ Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 + b_1 + c_1 = 1, a_2 + b_2 + c_2 = 0, a_3 + b_3 + c_3 = 0$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$2b_1 + c_1 = 0, 2b_2 + c_2 = 2, 2b_3 + c_3 = 0$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$c_1 = 0, c_2 = 0, c_3 = 2$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$b_1 = 0, b_2 = 1, b_3 = -1$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$a_1 = 1, a_2 = -1, a_3 = -1$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{vmatrix} = 2$$

114.(C) For in consistent solution

$$\Delta = 0, \text{ also any one of } \Delta_1, \Delta_2, \Delta_3 \neq 0$$

$$\begin{vmatrix} 2 & -4 & \lambda \\ 1 & -6 & 1 \\ \lambda & -10 & 4 \end{vmatrix} = 0 \Rightarrow 3\lambda^2 - 7\lambda - 6 = 0$$

$$(\lambda - 3)(3\lambda + 2) = 0$$

$$\lambda = 3, \lambda = \frac{-2}{3}$$

$$\text{For } \lambda = 3, \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$$\text{So in consistent only for } \lambda = \frac{-2}{3}$$

$$\mathbf{115.(C)} \quad \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & k^2 \\ 3 & 1 & 3 \end{vmatrix} = 0 \Rightarrow k^2 = 9$$

$$\text{So, } x + y + 3z = 0$$

$$x + 3y + 9z = 0$$

$$3x + y + 3z = 0 \text{ say } z = t \Rightarrow 2y + 6z = 0 \Rightarrow y = -3t$$

$$3x - 3t + 3t = 0 \Rightarrow x = 0 \Rightarrow x + \left(\frac{y}{z}\right) = -3$$

$$\mathbf{116.(D)} \quad \begin{vmatrix} x & a+y & x+a \\ y & b+y & y+b \\ z & c+y & z+c \end{vmatrix} c_3 \rightarrow c_3 - c_1 \quad \begin{vmatrix} x & a+y & a \\ y & b+y & b \\ z & c+y & c \end{vmatrix} c_2 = c_2 - c_3$$

$$\begin{vmatrix} x & y & a \\ y & y & b \\ z & y & c \end{vmatrix} = y \begin{vmatrix} x & 1 & a \\ y & 1 & b \\ z & 1 & c \end{vmatrix} = y(a-b)$$

$$\mathbf{117.(C)} \quad \Delta(x) = \begin{vmatrix} \cos^2 x & 1 + \sin^2 x & \sin 2x \\ 1 + \cos^2 x & \sin^2 x & \sin 2x \\ \cos^2 x & \sin^2 x & 1 + \sin 2x \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2$$

$$\Delta(x) = \begin{vmatrix} 2 & 1 + \sin^2 x & \sin 2x \\ 2 & \sin^2 x & \sin 2x \\ 1 & \sin^2 x & 1 + \sin 2x \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$$

$$\Delta(x) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & \sin^2 x & 1 + \sin 2x \end{vmatrix} = -1(2 + \sin 2x)$$

$$\Delta(x) = -\sin 2x - 2$$

$$-1 \leq \sin 2x \leq 1 \quad ; \quad -1 \leq -\sin 2x \leq 1 \quad ; \quad -3 \leq -\sin 2x - 2 \leq -1$$

$$\Delta(x) \in [-3, -1] \quad \therefore m = -3$$

$$M = -1$$

118.(B) $x + y + z = 2$

$$x + 2y + 3z = 5$$

$$x + 3y + \lambda z = \mu$$

For infinitely many solutions

$$D = 0 \quad D_1 = 0 \quad D_2 = 0 \quad \& \quad D_3 = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = 0$$

$$1(2\lambda - 9) + 1(3 - \lambda) + 1(3 - 2) = 0$$

$$2\lambda - 9 + 3 - \lambda + 1 = 0$$

$$\lambda - 5 = 0$$

$$\lambda = 5$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & \lambda & \mu \end{vmatrix} = 0$$

$$1(3\mu - 5\lambda) + 1(15 - 2\mu) + 2(2\lambda - 9) = 0$$

$$3\mu - 25 + 15 - 2\mu + 20 - 18 = 0$$

$$\mu - 10 + 2 = 0$$

$$\mu = 8$$

119.(A) $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$A^4 = \begin{bmatrix} \cos 4\theta & \sin 4\theta \\ -\sin 4\theta & \cos 4\theta \end{bmatrix}$$

$$B = A^4 + A = \begin{bmatrix} \cos 4\theta + \cos \theta & \sin 4\theta + \sin \theta \\ -(\sin 4\theta + \sin 2\theta) & \cos 4\theta + \cos \theta \end{bmatrix}$$

$$|B| = (\cos 4\theta + \cos \theta)^2 + (\sin 4\theta + \sin \theta)^2 = 2 + 2(\cos 4\theta \cos \theta + \sin 4\theta \sin \theta) = 2(1 + \cos 3\theta)$$

$$= 2 \left(1 + \cos \frac{3\pi}{5} \right) \quad \because \quad \theta = \frac{\pi}{5} = 2 \left(1 + \cos \left(\frac{\pi}{2} + \frac{\pi}{10} \right) \right) = 2 \left(1 - \sin \frac{\pi}{10} \right) \in (1, 2)$$

120.(3) For non-zero solutions $D = 0$

$$D = \begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$$

$$R_1 \rightarrow R_1 - R_3$$

$$R_2 \rightarrow R_2 - R_3$$

$$D = \begin{vmatrix} \lambda-3 & 0 & -(\lambda-3) \\ \lambda-3 & \lambda-3 & -2(\lambda-3) \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0 ; (\lambda-3)^2 \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$$

$$(\lambda-3)^2 [3\lambda-3+6\lambda+2-(3\lambda+1-2)] = 0$$

$$6\lambda(\lambda-3)^2 = 0 \Rightarrow \lambda = 0, 3$$

$$\text{Sum} = 0 + 3 = 3$$

121.(A) It is given that $a^3 + b^3 + c^3 = 2$

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \text{ and it is given } A^T \cdot A = I$$

$$A^T \cdot A = I \Rightarrow |A^T A| = |I|$$

$$|A^T| |A| = 1 \Rightarrow |A| |A| = 1 \Rightarrow |A| = \pm 1$$

$$|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$|A| = 3abc - a^3 - b^3 - c^3$$

$$\pm 1 = 3abc - 2$$

$$1 = 3abc - 2 \Rightarrow 3abc = 3 \Rightarrow abc = 1$$

$$-1 = 3abc - 2$$

$$3abc = 1$$

$$abc = \frac{1}{3}$$

122.(C) Here $P = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & -4 \\ 1 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$; $P = \begin{bmatrix} x+2y+z \\ -2x+3y-4z \\ x+9y-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x + 2y + z = 0 \dots\dots(i)$$

$$-2x + 3y - 4z = 0 \dots\dots(ii)$$

$$x + 9y - z = 0 \dots\dots(iii)$$

$$\frac{x}{-8-3} = \frac{y}{-2+4} = \frac{z}{3+4} = k ; x = -11k, y = 2k, z = 7k ;$$

$$\text{It is given that } x^2 + y^2 + z^2 = 1$$

$$121k^2 + 4k^2 + 49k^2 = 1 ; 174k^2 = 1 ; k = \pm \frac{1}{\sqrt{174}}$$

Solutions to Maths Workbook – 6 | Matrices and Determinants

JEE Advanced Archive

1.(A) If A and B are square matrices of equal degree, then $A + B = B + A$

2.(A) Every square matrix satisfied its characteristic equation,

$$\text{i.e. } |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\{(1-\lambda)(4-\lambda)+2\} \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow A^3 - 6A^2 + 11A - 6I = 0 \quad \dots\dots(i)$$

Given, $6A^{-1} = A^2 + cA + dI$, multiplying both sides by A, we get

$$6I = A^3 + cA^2 + dA \Rightarrow A^3 + cA^2 + dA - 6I = 0 \quad \dots\dots(ii)$$

On comparing equation (i) and (ii), we get $c = -6$ and $d = 11$

3.(A) Now, $P^T P = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \Rightarrow P^T P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P^T P = I \Rightarrow P^T = P^{-1}$

Since, $Q = PAP^T$

$$\begin{aligned} \therefore P^T Q^{2005} P &= P^T [(PAP^T)(PAP^T)\dots\dots 2005 \text{ times}] P \\ &= \underbrace{(P^T P)A(P^T P)A(P^T P)\dots(P^T P)A(P^T P)}_{2005 \text{ times}} = IA^{2005} = A^{2005} \end{aligned}$$

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\dots\dots\dots$$

$$A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

$$\therefore P^T Q^{2005} P = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

4.(D) Given, $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix}$

Also, given, $A^2 = B$

$$\Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \Rightarrow \alpha^2 = 1 \text{ and } \alpha+1 = 5$$

Which is not possible at the same time

$\therefore$ No real values of α exists.

5.(D) PLAN : It is a simple question on scalar multiplication, i.e.

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Description of Situation construction of matrix

$$\text{i.e. if } a = [a_{ij}]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Here, } P = [a_{ij}]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Q = [b_{ij}]_{3 \times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Where, $b_{ij} = 2^{i+j} a_{ij}$

$$\begin{aligned} \therefore |Q| &= \begin{vmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{vmatrix} \\ &= 4 \times 8 \times 16 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{vmatrix} = 2^9 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2^{12} \cdot |P| = 2^{12} \cdot 2^{13} \end{aligned}$$

6.(C) We know, $|A^n| = |A|^n$

$$\text{Since, } |A^3| = 125 \Rightarrow |A|^3 = 125 \Rightarrow \begin{vmatrix} \alpha & 2 \\ 2 & \alpha \end{vmatrix} = 5 \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$$

$$7.(C) \text{ Given, } \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$

$$\begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix} = (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$

$$\Rightarrow (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$$\Rightarrow (2\cos x + \sin x)(\sin x - \cos x)^2 = 0$$

$$\Rightarrow 2\cos x + \sin x = 0 \text{ or } \sin x - \cos x = 0$$

$$\Rightarrow 2\cos x = -\sin x \text{ or } \sin x = \cos x$$

$$\Rightarrow \cot x = -1/2 \text{ gives no solution in } -\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \text{ and } \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$$

8.(A) Given, $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$

Applying $C_3 \rightarrow C_3 - (C_1 + C_2)$

$$= \begin{vmatrix} 1 & x & 0 \\ 2x & x(x-1) & 0 \\ 3x(x-1) & x(x-1)(x-2) & 0 \end{vmatrix} = 0$$

$$\therefore f(x) = 0 \Rightarrow f(100) = 0$$

9.(B) Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 + C_1 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ \cos(p-d)x + \cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x + \sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix} \Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ 2\cos px \cos dx & \cos px & \cos(p+d)x \\ 2\sin px \cos dx & \sin px & \sin(p+d)x \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - 2\cos dx \cdot C_2$

$$\Rightarrow \Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

$$\Rightarrow \Delta = (1+a^2-2a\cos dx) [\sin(p+d)x \cos px - \sin px \cos(p+d)x] \Rightarrow \Delta = (1+a^2-2a\cos dx) \sin dx$$

Which is independent of p .

10.(B) Given, $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$

Applying $C_1 \rightarrow C_1 - (pC_2 + C_3)$

$$\Rightarrow \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2+yp+yp+z) & xp+y & yp+z \end{vmatrix} = 0$$

$$\Rightarrow -(xp^2+2yp+z)(xz-y^2) = 0$$

$$\therefore \text{Either } xp^2+2yp+z=0 \text{ or } y^2=xz$$

$$\Rightarrow x, y, z \text{ are in GP.}$$

11.(B) Since, A is the determinant of order 3 with entries 0 or 1 only.

Also, B is the subset of A consisting of all determinants with value 1

[since, if we interchange any two rows or columns, then among themselves sign changes]

Given, C is the subset having determinant with value -1.

$\therefore$ B has as many elements as c.

12.(B) PLAN Use the following properties of transpose $(AB)^T = B^T A^T, (A^T)^T = A$ and $A^{-1}A = I$ and simplify. If A is non-singular matrix, then $|A| \neq 0$.

$$\begin{aligned}
 \text{Given, } AA^T &= A^T A \text{ and } B = A^{-1}A^T \\
 BB^T &= (A^{-1}A^T)(A^{-1}A^T)^T \\
 &= A^{-1}A^T A(A^{-1})^T \quad [\because (AB)^T = B^T A^T] \\
 &= A^{-1}AA^T(A^{-1})^T \quad [\because AA^T = A^T A] \\
 &= IA^T(A^{-1})^T \quad [\because A^{-1}A = I] \\
 &= A^T(A^{-1})^T = (A^{-1}A)^T \quad [\because (AB)^T = B^T A^T] \\
 &= I^T = I
 \end{aligned}$$

13.(B) Given, $P^T = 2P + I$... (i)

$$\begin{aligned}
 \therefore (P^T)^T &= (2P + I)^T = 2P^T + I \\
 \Rightarrow P &= 2P^T + I \Rightarrow P = 2(2P + I) + I \\
 \Rightarrow P &= 4P + 3I \quad \text{or} \quad 3P = -3I \Rightarrow PX = -IX = X
 \end{aligned}$$

14.(A) $|A| \neq 0$, as non-singular

$$\begin{aligned}
 \therefore \begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} &\neq 0 \\
 \Rightarrow 1(1 - c\omega) - a(\omega - c\omega^2) + b(\omega^2 - \omega^2) &\neq 0 \Rightarrow 1 - c\omega - a\omega + ac\omega^2 \neq 0 \Rightarrow (1 - c\omega)(1 - a\omega) \neq 0 \\
 \Rightarrow a \neq \frac{1}{\omega}, c \neq \frac{1}{\omega} &\Rightarrow a = \omega, c = \omega \text{ and } b \in \{\omega, \omega^2\} \Rightarrow 2 \text{ solutions}
 \end{aligned}$$

15.(B) Given, $M^T = -M, N^T = -N$ and $MN = NM$... (i)

$$\begin{aligned}
 \therefore M^2 N^2 (M^T N)^{-1} (MN^{-1})^T \\
 \Rightarrow M^2 N^2 N^{-1} (M^T)^{-1} (N^{-1})^T \cdot M^T &\Rightarrow M^2 N (NN^{-1}) (-M)^{-1} (N^T)^{-1} (-M) \\
 \Rightarrow M^2 N I (-M^{-1}) (-N)^{-1} (-M) &\Rightarrow -M^2 NM^1 N^{-1} M \\
 \Rightarrow -M \cdot (MN) M^{-1} N^{-1} M &= -(M(NM) M^{-1} N^{-1} M) \\
 \Rightarrow -MN(NM^{-1}) N^{-1} M &= -M(NN^{-1}) M \Rightarrow -M^2
 \end{aligned}$$

NOTE Here, non-singular word should not be used, since there is no non-singular 3×3 skew-symmetric matrix.

16.(A) Since, $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is linear equation in three variables and that could have only unique, no solution or

infinitely many solution.

$\therefore$ It is not possible to have two solutions.

Hence, number of matrices A is zero.

17.(B) Since, given system has no solution.

$\therefore \Delta = 2$ and any one amongst $\Delta_x, \Delta_y, \Delta_z$ is non-zero.

$$\text{Let } \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0 \text{ and } \Delta_z = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -4 \\ 1 & 1 & 4 \end{vmatrix} = 6 \neq 0 \Rightarrow \lambda = 1$$

18.(B) For infinitely many solutions, we must have $\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1} \Rightarrow k = 1$

19.(D) Since, the given system has non-zero solution.

$$\therefore \begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 + C_3$

$$\Rightarrow \begin{vmatrix} 1+k & -k-1 & -1 \\ 1+k & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 0 \Rightarrow 2(k+1) - (k+1)^2 = 0 \Rightarrow (k+1)(2-k-1) = 0 \Rightarrow k = \pm 1$$

NOTE There is a golden rule in determinant that n one's $\Rightarrow (n-1)$ zero's or n (constant) $\Rightarrow (n-1)$ zero's for all constant should be in a single row or a single column.

20.(A) The given system of equations can be expressed as

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ k \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 + R_1$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ k-1 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ k-3 \end{bmatrix}$$

When $k \neq 3$, the given system of equations has no solution.

$\Rightarrow$ Statement I is true. Clearly, Statement II is also true as it is rearrangement of rows and columns

$$\text{of } \begin{bmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{bmatrix}.$$

$$\mathbf{21.(B)} \quad P \equiv \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}; \quad P^2 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix}; \quad P^3 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 96 & 12 & 1 \end{bmatrix}; \quad P^4 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 16 & 1 & 0 \\ 160 & 16 & 1 \end{bmatrix}$$

Pattern of element P_{31} is $16[1, 3, 6, 10, \dots]$

$\therefore$ 50th term is 16×1275

[By observing that T_n of $S = 1 + 3 + 6 + 10 + \dots$ is $\frac{n^2+n}{2}$]

$$\therefore P^{50} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 16 \times 1275 & 200 & 1 \end{bmatrix}; \quad Q = P^{50} - I = \begin{bmatrix} 0 & 0 & 0 \\ 200 & 0 & 0 \\ 16 \times 1275 & 200 & 0 \end{bmatrix}$$

$$\therefore \frac{q_{31} + q_{32}}{q_{21}} = \frac{16 \times 1275 + 200}{200} = 102 + 1 = 103$$

$$22.(B) \quad M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}; \quad M^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$M^T M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$T_r(M^T M) = (a_1^2 + b_1^2 + c_1^2) + (a_2^2 + b_2^2 + c_2^2) + (a_3^2 + b_3^2 + c_3^2) = 5$$

$$5 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 \Rightarrow 5, 1^s, 4, 0^s \Rightarrow {}^9C_5$$

$$1^2 + 2^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 \Rightarrow 1 \rightarrow 1, 1 \rightarrow 2, 7 \rightarrow 0^s \Rightarrow {}^9C_7 \times {}^2C_1$$

$$\Rightarrow 126 + \frac{9.8}{2} \times 2 = 198$$

23.(CD)(a) $(N^T M N)^T = N^T M^T (N^T)^T = N^T M^T N$, is symmetric if M is symmetric and skew-symmetric, if M is skew-symmetric.

$$(b) \quad (MN - NM)^T = (MN)^T - (NM)^T \\ = NM - MN = -(MN - NM)$$

$\therefore$ Skew-symmetric, when M and N are symmetric.

$$(c) \quad (MN)^T = N^T M^T = NM \neq MN$$

$\therefore$ Not correct.

$$(d) \quad (adj MN) = (adj N) \cdot (adj M) \quad \therefore \text{Not correct.}$$

24.(AB) PLAN

(i) If A and B are two non-zero matrices and $AB = BA$, then $(A - B)(A + B) = A^2 - B^2$.

(ii) The determinant of the product of the matrices is equal to product of their individual determinants, i.e. $|AB| = |A| |B|$

$$\text{Given, } M^2 = N^4 \Rightarrow M^2 - N^4 = 0$$

$$\Rightarrow (M - N^2)(M + N^2) = 0 \quad [\text{as } MN = NM]$$

$$\text{Also, } M \neq N^2 = 0$$

$$\Rightarrow M + N^2 = 0$$

$$\Rightarrow \det(M + N^2) = 0$$

$$\text{Also, } \det(M^2 + MN^2) = (\det M)(\det M + N^2) = (\det M)(0) = 0$$

$$\text{As, } \det(M^2 + MN^2) = 0$$

Thus, there exists a non-zero matrix U such that $(M^2 + MN^2)U = 0$

25.(BCD) Here, $P = [p_{ij}]_{n \times n}$ with $p_{ij} = \omega^{i+j}$

$\therefore$ When $n = 1$

$$P[p_{ij}]_{1 \times 1} = [\omega^2] \Rightarrow P^2 = [\omega^4] \neq 0$$

$\therefore$ When $n = 2$

$$P = [p_{ij}]_{2 \times 2} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} \omega^2 & \omega^3 \\ \omega^3 & \omega^4 \end{bmatrix} = \begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix} \begin{bmatrix} \omega^2 & 1 \\ 1 & \omega \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} \omega^4 + 1 & \omega^2 + \omega \\ \omega^2 + \omega & 1 + \omega^2 \end{bmatrix} \neq 0$$

When $n = 3$

$$P = [p_{ij}]_{3 \times 3} = \begin{bmatrix} \omega^2 & \omega^3 & \omega^4 \\ \omega^3 & \omega^4 & \omega^5 \\ \omega^4 & \omega^5 & \omega^6 \end{bmatrix} = \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix} \begin{bmatrix} \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore P^2 = 0$, when n is a multiple of 3

$P^2 \neq 0$, when n is not a multiple of 3

$\Rightarrow n = 57$ is not possible

$\therefore n = 55, 58, 56$ is possible.

26.(BD) Given,

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$$

Applying $C_3 \rightarrow C_3 - (\alpha C_1 + C_2)$

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix} = 0$$

$$\Rightarrow -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$$

$$\Rightarrow a\alpha^2 + 2b\alpha + c = 0 \text{ or } b^2 = ac$$

$$\Rightarrow x - \alpha \text{ is a factor of } ax^2 + 2bx + c \text{ or } a, b, c \text{ are in GP.}$$

27.(CD) PLAN A square matrix M is invertible, iff $\det(M) \neq 0$ or $|M| \neq 0$.

Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(A) Given, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} \Rightarrow a = b = c = \alpha$ [let]

$$\Rightarrow M = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} \Rightarrow |M| = 0 \Rightarrow M \text{ is non-invertible}$$

(B) Given, $[b \ c] = [a \ b]$

$$\Rightarrow a = b = c = \alpha \quad [\text{let}]$$

Again, $|M| = 0$

$\Rightarrow M$ is non-invertible.

(C) As given $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$ $[\because a \text{ and } c \text{ are non-zero}]$

$\Rightarrow M$ is invertible.

(D) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow |M| = ac - b^2 \neq 0$

$\therefore ac$ is not equal to square of an integer.
 M is invertible.

28.(AD) PLAN If $|A_{n \times n}| = \Delta$, then $|adj A| = \Delta^{A-1}$

$$\text{Here, } adj P_{3 \times 3} = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow |adj P| = |P|^2$$

$$\therefore |adj P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 1(3-7) - 4(6-7) + 4(2-1) = -4 + 4 + 4 = 4 \Rightarrow |P| = \pm 2$$

29.(CD) $X_{3 \times 3}, Y_{3 \times 3} \quad X, Y \neq 0 \quad X' = -X, Y' = -Y$

$$|X| = 0 = |Y|$$

$$Z_{3 \times 3} \neq 0, Z' = Z$$

$$\begin{aligned} (Y^3 Z^4 - Z^4 Y^3)' &= (Y^3 Z^4)' - (Z^4 Y^3)' = (Z')^4 (Y')^3 - (Y')^3 (Z')^4 \\ &= -Z^4 Y^3 + Y^3 Z^4 = Y^3 Z^4 - Z^4 Y^3 \quad \text{Symmetric} \end{aligned}$$

$$(X^{44} + Y^{44})' = (X')^{44} + (Y')^{44} = X^{44} + Y^{44} \quad \text{Symmetric}$$

$$\begin{aligned} (X^4 Z^3 - Z^3 X^4)' &= (Z')^3 (X')^4 - (X')^4 (Z')^3 = Z^3 X^4 - X^4 Z^3 \\ &= -(X^4 Z^3 - Z^3 X^4) \quad \text{Skew symmetric} \end{aligned}$$

$$(X^{23} + Y^{23})' = (X')^{23} + (Y')^{23} = -X^{23} - Y^{23} = -(X^{23} + Y^{23}) \quad \text{Skew Symmetric}$$

$$30.(BC) \begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (1+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = \begin{vmatrix} 1+2\alpha+\alpha^2 & 1+4\alpha+4\alpha^2 & 1+6\alpha+9\alpha^2 \\ 4+4\alpha+\alpha^2 & 4+8\alpha+4\alpha^2 & 4+12\alpha+9\alpha^2 \\ 9+6\alpha+\alpha^2 & 9+12\alpha+4\alpha^2 & 9+18\alpha+9\alpha^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 4 & 2\alpha & \alpha^2 \\ 9 & 3\alpha & \alpha^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 7 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = \alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 4 \\ 1 & 3 & 8 \end{vmatrix} = \alpha^3 \begin{vmatrix} 1 & 0 & 0 \\ 4 & -2 & -3 \\ 9 & -6 & -8 \end{vmatrix} \times 4 = -8\alpha^3$$

$$-8\alpha^3 = -648\alpha \Rightarrow \alpha(\alpha^2 - 81) = 0 \Rightarrow \alpha = 0, \pm 9$$

31.(BC) $Q = KP^{-1}$

$$P^{-1} = \frac{\begin{bmatrix} 5\alpha & 10 & -\alpha \\ 3\alpha & 6 & -(3\alpha+4) \\ -10 & 12 & 2 \end{bmatrix}}{20+12\alpha}$$

$$q_{23} \equiv -\frac{k}{8}$$

$$\frac{k(-3\alpha-4)}{20+12\alpha} \equiv \frac{-k}{8}$$

$$\alpha = -1$$

$$|Q| \equiv \frac{k^2}{2}$$

$$|kP^{-1}| = \frac{k^2}{2} \quad |P| = 8 \quad |Q| = 8$$

$$\frac{k^3}{|P|} = \frac{k^2}{2}$$

$$\frac{k^3}{20+12\alpha} = \frac{k^2}{2}$$

$$k = 4$$

$$4\alpha - k + 8 = 0 \quad \text{Ans. B}$$

$$|P_{adj} \cdot Q| = |P||Q|^2 \equiv 2^3 \cdot 2^6 \equiv 2^9 \quad \text{Ans. C}$$

32.(BCD) $|A| = \begin{vmatrix} a & 2 \\ 3 & -2 \end{vmatrix} = -2a - 6 = -2(a+3)$

$$|A_1| = \begin{vmatrix} \lambda & 2 \\ \mu & -2 \end{vmatrix} = -2(\lambda + \mu)$$

$$|A_2| = \begin{vmatrix} a & \lambda \\ 3 & \mu \end{vmatrix} = a\mu - 3\mu$$

For $a = -3$ $|A| = 0$ system as infinite many solutions and if $|A_1| = 0$ and $|A_2| = 0$

$\Rightarrow \lambda + \mu = 0 \Rightarrow$ Option A is incorrect

If $\lambda + \mu$ and $a = -3$

$\Rightarrow |A| = 0, |A_2| = 0, |A_1| = 0 \Rightarrow$ System has infinite solutions

For $\lambda + \mu \neq 0$ and $a = -3 \Rightarrow |A| = 0, |A_2| \neq 0, |A_1| \neq 0$

$\Rightarrow$ System has no solution $\Rightarrow$ Option D is correct

33.(AB) For (A) and (B) $\det(A^2) = -1$,

and for (C)

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1 = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\text{For (C): } I^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$34.(AD) \quad \Delta = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -4 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 0$$

$\therefore$ For atleast one solution, must be infinite solution ($\because \Delta = 0$)

$$\therefore \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$$\Delta_1 = \begin{vmatrix} b_1 & 2 & 5 \\ b_2 & -4 & 3 \\ b_3 & -2 & 2 \end{vmatrix} = 0$$

$$b_1 + 7b_2 = 13b_3 : S$$

$$(A) \quad \Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 2 & 6 \end{vmatrix} = 12 \quad (\text{Unique solution})$$

$$(B) \quad \Delta = \begin{vmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{vmatrix} = 0$$

$$\Delta_1 = \begin{vmatrix} b_1 & 1 & 3 \\ b_2 & 2 & 6 \\ b_3 & -1 & -3 \end{vmatrix} = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & b_1 & 3 \\ 5 & b_2 & 6 \\ -2 & b_3 & -3 \end{vmatrix} \equiv 3(b_1 + b_2 + 3b_3)$$

which will not be zero for each b_1, b_2, b_3 in S

$\therefore$ No solution

$$(C) \quad \Delta = \begin{vmatrix} -1 & 2 & -5 \\ 2 & -4 & 10 \\ 1 & -2 & 5 \end{vmatrix} = 0$$

$$-x + 2y - 5z = b_1 \quad \dots(i)$$

$$2x - 4y + 10z = b_2 \quad \dots(ii)$$

$$x - 2y + 5z = b_3 \quad \dots(iii)$$

$$(ii) + 2(i)$$

$$0 = 2b_1 + b_2$$

$$(iii) + (i)$$

$$0 = b_1 + b_3$$

$$\begin{bmatrix} -x + 2y - 5z : & b_1 \\ 0 & : & 2b_1 + b_2 \\ 0 & : & b_1 + b_3 \end{bmatrix}$$

For a solution

$$2b_1 + b_2 = 0$$

$$b_1 + b_3 = 0$$

$$b_1 = \frac{-b_2}{2}$$

$$b_1 = -b_3$$

$\therefore$ For every $b_1, b_2, b_3 \in S$ infinite solution will not exist

$$(D) \Delta = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & -5 \end{vmatrix} = 54 \text{ unique solution}$$

$$35.(C) \quad M = \alpha I + \beta M^{-1}$$

$$M^2 - \alpha M - \beta I = 0 \quad \Rightarrow \quad \alpha = T_r(M) = \sin^4 \theta + \cos^4 \theta \in \left[\frac{1}{2}, 1 \right]$$

$$\beta = -|M| = -(S^4 C^4 + (1 + S^2)(1 + C^2)) = -(S^4 C^4 + S^2 C^2 + 2), S^2 C^2 = t$$

$$= -(t^2 + t + 2) \quad t \in \left[0, \frac{1}{4} \right]$$

$$\alpha^* = \frac{1}{2}, \beta^* = -\frac{37}{16} \Rightarrow \alpha^* + \beta^* = -\frac{29}{16}$$

$$36.(ABC) \quad \text{adj.}(\text{adj. } M) = |M|^2 \quad a = 2, b = 1$$

$$= \begin{bmatrix} 0 & -2 & -4 \\ -2 & -4 & -6 \\ -6 & -2 & -2 \end{bmatrix} = -2 \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \Rightarrow |M| = -2$$

$$\beta + 2\gamma = 1 \Rightarrow \beta = 1 - 2\gamma$$

$$\alpha + 2\beta + 3\gamma = 2 \Rightarrow \alpha = 2 - 2\beta - 3\gamma = 2 - 2 + 4\gamma - 3\gamma = \gamma$$

$$3\alpha + \beta + \gamma = 3 \Rightarrow 3\gamma + 1 - 2\gamma + \gamma = 3$$

$$2\gamma = 2$$

$$\gamma = 1$$

$$\beta = 1 - 2 = -1$$

$$\alpha = 1$$

$$\alpha - \beta + \gamma = 1 + 1 + 1 = 3$$

$$(\text{adj. } M)^{-1} + \text{adj. } M^{-1} = 2(\text{adj. } M)^{-1} = \frac{2}{|M|} M = -M$$

$$|\text{adj. } M^2| = |\text{adj. } M|^2 = |M|^4 = (-2)^4 = 16$$

$$37.(ABD) \text{ Clearly } P_K = P_K^T = P_K^{-1}$$

As well as $A^T = A$ where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\text{Given } X = \sum_{k=1}^6 P_K A P_K^T$$

$$\Rightarrow X^T = \sum_{k=1}^6 P_K A^T P_K^T \Rightarrow X^T = \sum_{k=1}^6 P_K A P_K^T \quad [\text{As } A^T = A]$$

$$\Rightarrow X^T = X$$

$$\text{Let } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow XB = \sum_{K=1}^6 P_K A P_K^T B = \sum_{K=1}^6 P_K A P_K B \quad [\text{As } P_K^T = P_K]$$

$$\Rightarrow XB = \sum_{k=1}^6 P_K A B = (P_1 + P_2 + \dots + P_6) \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = 30B \Rightarrow \alpha = 30$$

$$\Rightarrow (X - 30I) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\text{As } (X - 30I) B = 0 \text{ has a non trivial solution } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \det(X - 30I) = 0$$

$$\Rightarrow X - 30I \text{ is non invertible } T_r(X) = 3 \times 6 = 18$$

$$38.(AB) \det R = \det(PQP^{-1})$$

$$\Rightarrow \det R = \det Q$$

$$\Rightarrow \det R = \begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x+0 & x+0 & 5+1 \end{vmatrix} = \begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{vmatrix} + \begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \det R = \begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{vmatrix} + 8 \quad \forall x \in R$$

$$\text{As } P^{-1} = \frac{1}{6} \begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 2+x & \frac{x}{2}+1 & \frac{2}{3}-\frac{x}{3} \\ 2x & 4 & \frac{4}{3}+\frac{2}{3}x \\ 3x & 0 & 6-x \end{bmatrix}$$

$$\text{As for } x=1 \det R \neq 0$$

$$\text{So, } R \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = \beta = \gamma = 0$$

$$\text{As } R \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} \quad \text{or} \quad (R - 6I) \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 0 \Rightarrow a = 2, b = 3$$

$$\Rightarrow PQ = QP$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix} = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Comparing a_{12}

$$\Rightarrow 2x + 4 = 2x + 2 \Rightarrow x \in \phi$$

$$39.(0) \quad \text{Let } \Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$$

On dividing and multiplying R_1, R_2, R_3 by $\log x, \log y, \log z$, respectively

$$= \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

$$40.(0) \quad \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$\text{Now, } \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

Applying $R_1 \rightarrow aR_1, R_2 \rightarrow bR_2, R_3 \rightarrow cR_3$

$$= \frac{1}{abc} \cdot abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$$

41.(2 and 7)

$$\text{Given, } \begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_2 + R_3$

$$\Rightarrow \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow (x+9) \begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$

$$\Rightarrow (x+9) \begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix} = 0 \Rightarrow (x+9)(x-2)(x-7) = 0$$

$\Rightarrow x = -9, 2, 7$ are the roots.

$\therefore$ Other two roots are 2 and 7.

42. {-1, 2}

$$\text{Given, } \begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$

$$\Rightarrow 1(-10x^2 - 10x) - 4(5x^2 - 5) + 20(2x + 2) = 0$$

$$\Rightarrow -30x^2 + 30x + 60 = 0$$

$$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2, -1$$

Hence, the solution set is $\{-1, 2\}$.

$$\text{43.(0) Given, } \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix} = p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$$

Thus, the value of t is obtained by putting $\lambda = 0$.

$$\Rightarrow \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = t \Rightarrow t = 0 \quad [\because \text{determinants of odd order skew-symmetric matrix is zero}]$$

$$44.(4) \quad |A| = (2k+1)^3, |B| = 0$$

$$\text{But } \det(\text{adj } A) + \det(\text{adj } B) = 10^6 \Rightarrow (2k+1)^6 = 10^6 \Rightarrow k = \frac{9}{2} \Rightarrow [k] = 4$$

$$45. \lambda = 0$$

Given system $\lambda x + y + z = 0, -x + \lambda y + z = 0$

and $-x - y + \lambda z = 0$

will have non-zero solution, if

$$\begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2 + 1) - 1(-\lambda + 1) + 1(1 + \lambda) = 0 \Rightarrow \lambda^3 + \lambda + \lambda - 1 + 1 + \lambda = 0$$

$$\Rightarrow \lambda^3 + 3\lambda = 0 \Rightarrow \lambda(\lambda^2 + 3) = 0 \Rightarrow \lambda = 0$$

46. Let $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$\therefore M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad M \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} a_1 - a_2 \\ b_1 - b_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$$\Rightarrow a_2 = -1, b_2 = 2, c_2 = 3, a_1 - a_2 = 1, b_1 - b_2 = 1, c_1 - c_2 = -1$$

$$\Rightarrow a_1 + a_2 + a_3 = 0, b_1 + b_2 + b_3 = 0$$

$$c_1 + c_2 + c_3 = 12$$

$$\therefore a_1 = 0, b_2 = 2, c_3 = 7$$

$$\Rightarrow \text{Sum of diagonal elements} = 0 + 2 + 7 = 9$$

47.(1) $z \equiv -\frac{1 + \sqrt{3}i}{2}$

$$z \equiv \omega$$

$$P = \begin{bmatrix} (-\omega)^r & (\omega^2)^s \\ (\omega^2)^s & \omega^r \end{bmatrix}$$

$$P^2 = -I$$

$$\begin{bmatrix} (-\omega)^r & (\omega^2)^s \\ (\omega^2)^s & \omega^r \end{bmatrix} \begin{bmatrix} (-\omega)^r & (\omega^2)^s \\ (\omega^2)^s & \omega^r \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s} \cdot (1 + (-1)^r) \\ \omega^{r+2s} (1 + (-1)^r) & \omega^{4s} + \omega^{2r} \end{bmatrix} \equiv \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\omega^{2r} + \omega^{4s} \equiv -1 \text{ and } \omega^{r+2s} (1 + (-1)^r) = 0 \text{ } r \text{ can be 1 or 3}$$

$$\text{If } r \equiv 1$$

$$\omega^2 + \omega^{4s} = -1$$

$$\omega^{4s} \equiv -1 - \omega^2$$

$$\omega^{4s} \equiv \omega \Rightarrow s = 1$$

$$\text{If } r = 3$$

$$\omega^6 + \omega^{4s} = -1$$

$$\omega^{4s} \equiv -2$$

No "s" possible

Ordered pair of (r, s) is $(1, 1)$

$$\begin{aligned}
 48.(2) \quad & \begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x & 1+27x^3 \end{vmatrix} \equiv 10 \\
 & \begin{vmatrix} x & x^2 & 1 \\ 2x & 4x^2 & 1 \\ 3x & 9x^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 2x & 4x^2 & 8x^3 \\ 3x & 9x^2 & 27x^3 \end{vmatrix} \equiv 10 \\
 & \begin{vmatrix} 1 & x & x^2 \\ 1 & 2x & 4x^2 \\ 1 & 3x & 9x^2 \end{vmatrix} + x \cdot 2x \cdot 3x \begin{vmatrix} 1 & x & x^2 \\ 1 & 2x & 4x^2 \\ 1 & 3x & 9x^2 \end{vmatrix} \equiv 10 \\
 & -x \cdot -x \cdot 2x (1+6x^3) \equiv 10 \\
 & 2x^3(1+6x^3) \equiv 10 \\
 & x^3(1+6x^3) \equiv 5 \\
 & x^3 = t \\
 & t + 6t^2 \equiv 5 \\
 & 6t^2 + t - 5 = 0 \\
 & 6t^2 + 6t - 5t - 5 = 0 \\
 & 6t(t+1) - 5(t+1) = 0 \\
 & t \equiv -1, t = 5/6 \\
 & x^3 = -1, x^3 = 5/6 \\
 & x = -1 \text{ and } x \equiv (5/6)^{1/3}
 \end{aligned}$$

$$49.(1) \quad \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & 1 & \alpha \\ 1 & \alpha & 1 \end{vmatrix} = (\alpha^2 - 1)^2 = 0$$

$$\alpha^2 = 1$$

$$\alpha = \pm 1$$

$\alpha = 1$ (two planes are parallel) (Rejected)

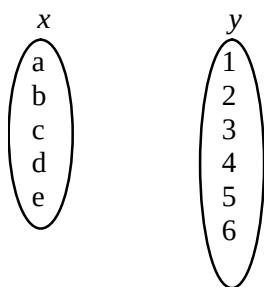
$\alpha = -1$ (two planes are coincident)

50.(4.00)

For maximum value of determinant we can make

Diagonal entries minimum and non-diagonal entries maximum

$$\Delta_{\max} = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4$$



51.(D) As (a, b, c) lies on $2x + y + z = 1 \Rightarrow 2a + b + c = 1$

$$\Rightarrow 2a + 6a - 7a = 1 \Rightarrow a = 1, b = 6, c = -7$$

$$\therefore 7a + b + c = 7 + 6 - 7 = 6$$

52.(B) If $b = 6 \Rightarrow a = 1$ and $c = -7$

$$\therefore ax^2 + bx + c = 0 \Rightarrow x^2 + 6x - 7 = 0$$

$$\Rightarrow (x + 7)(x - 1) = 0$$

$$\therefore x = 1, -7$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{1} - \frac{1}{7} \right)^n = 1 + \frac{6}{7} + \left(\frac{6}{7} \right)^2 + \dots + \infty = \frac{1}{1 - \frac{6}{7}} = \frac{1}{1/7} = 7$$

53.(A) If $a = 2, b = 12, c = -14$

$$\therefore \frac{3}{\omega^2} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$$

$$\Rightarrow \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}} = \frac{3}{\omega^2} + 1 + 3\omega^2 = 3\omega + 1 + 3\omega^2 = 1 + 3(\omega + \omega^2) = 1 - 3 = -2$$

54.(D) The number of matrices for which p does not divide $\text{Tr}(A) = (p-1)p^2$ of these $(p-1)^2$ are such that p divides $|A|$. The number of matrices for which p divides $\text{Tr}(A)$ and p does not divide $|A|$ are $(p-1)^2$.

$$\therefore \text{Required number} = (p-1)p^2 - (p-1)^2 + (p-1)^2 = p^3 - p^2$$

55.(C) Trace of $A = 2a$, will be divisible by p , iff $a = 0$.

$|A| = a^2 - bc$, for $(a^2 - bc)$ to be divisible by p . There are exactly $(p-1)$ ordered pairs (b, c) for any value of a .

$$\therefore \text{Required number is } (p-1)^2$$

56.(D) Given, $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$, $a, b, c \in \{0, 1, 2, \dots, p-1\}$

If A is skew-symmetric matrix, then $a = 0, b = -c$

$$\therefore |A| = -b^2$$

Thus, P divides $|A|$, only when $b = 0$ (i)

Again, if A is symmetric matrix, then $b = c$ and $|A| = a^2 - b^2$

Thus, p divides $|A|$, if either p divides $(a-b)$ or p divides $(a+b)$

p divides $(a-b)$, only when $a = b$, i.e. $a - b \in \{0, 1, 2, \dots, (p-1)\}$ i.e. p choices(ii)

p divides $(a + b)$

$\Rightarrow$ p choices, including $a = b = 0$ included in equation (i)

$\therefore$ Total number of choices are $(p + p - 1) = 2p - 1$

57. Since, $M^T M = I$ and $|M| = 1$

$$\therefore |M - I| = |IM - M^T M| \quad [\because IM = M]$$

$$\Rightarrow |M - I| = |(I - M^T)M| = |(I - M)^T| \quad |M| = |I - M|$$

$$= (-1)^3 |M - I| \quad [\because I - M \text{ is a } 3 \times 3 \text{ matrix}]$$

$$= -|M - I|$$

$$\Rightarrow 2|M - I| = 0 \Rightarrow |M - I| = 0$$

58.(4) Given $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc = 1$ and $A^T A = I$ (i)

Now, $A^T A = I$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ab + bc + ca \\ ab + bc + ca & a^2 + b^2 + c^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2 + b^2 + c^2 = 1 \text{ and } ab + bc + ca = 0 \quad \text{.....(ii)}$$

We know, $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$$\Rightarrow a^3 + b^3 + c^3 = (a + b + c)(1 - 0) + 3 \quad (\text{from Equations (i) and (ii)})$$

$$\therefore a^3 + b^3 + c^3 = (a + b + c) + 3 \quad \text{.....(iii)}$$

$$\text{Now, } (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1 \quad \text{.....(iv)}$$

$$\text{From equation (iii), } a^3 + b^3 + c^3 = 1 + 3 \Rightarrow a^3 + b^3 + c^3 = 4$$

59. Given, $\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{a} \begin{vmatrix} a^2x - aby - ac & bx + ay & cx + a \\ abx + a^2y & -ax + by - c & cy + b \\ acx + a^2 & cy + b & -ax - by + c \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 + bC_2 + cC_3$

$$\Rightarrow \frac{1}{a} \begin{vmatrix} (a^2 + b^2 + c^2)x & ay + bx & cx + a \\ (a^2 + b^2 + c^2)y & by - c - ax & b + cy \\ a^2 + b^2 + c^2 & b + cy & c - ax - by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{a} \begin{vmatrix} x & ay + bx & cx + a \\ y & by - c - ax & b + cy \\ 1 & b + cy & c - ax - by \end{vmatrix} = 0 \quad (\because a^2 + b^2 + c^2 = 1)$$

Applying $C_2 \rightarrow C_2 - bC_1$ and $C_3 \rightarrow C_3 - cC_1$

$$\Rightarrow \frac{1}{a} \begin{vmatrix} x & ay & a \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{ax} \begin{vmatrix} x^2 & axy & ax \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_1 + yR_2 + R_3$

$$\Rightarrow \frac{1}{ax} \begin{vmatrix} x^2 + y^2 + 1 & 0 & 0 \\ y & -c - ax & b \\ 1 & cy & -ax - by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{ax} [(x^2 + y^2 + 1)((-c - ax)(-ax - by) - b(cy))] = 0$$

$$\Rightarrow \frac{1}{ax} [(x^2 + y^2 + 1)(acx + bcy + a^2x^2 + abxy - bcy)] = 0$$

$$\Rightarrow \frac{1}{ax} [(x^2 + y^2 + 1)(acx + a^2x^2 + abxy)] = 0$$

$$\Rightarrow \frac{1}{ax} [ax(x^2 + y^2 + 1)(c + ax + by)] = 0$$

$$\Rightarrow (x^2 + y^2 + 1)(ax + by + c) = 0$$

$$\Rightarrow ax + by + c = 0$$

Which represents a straight line.

60. Let $\Delta = \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ \sin \left(\theta + \frac{2\pi}{3} \right) & \cos \left(\theta + \frac{2\pi}{3} \right) & \sin \left(2\theta + \frac{4\pi}{3} \right) \\ \sin \left(\theta - \frac{2\pi}{3} \right) & \cos \left(\theta - \frac{2\pi}{3} \right) & \sin \left(2\theta - \frac{4\pi}{3} \right) \end{vmatrix}$

Applying $R_2 \rightarrow R_2 + R_3$

$$= \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ \sin \left(\theta + \frac{2\pi}{3} \right) & \cos \left(\theta + \frac{2\pi}{3} \right) & \sin \left(2\theta + \frac{4\pi}{3} \right) \\ + \sin \left(\theta - \frac{2\pi}{3} \right) & + \cos \left(\theta - \frac{2\pi}{3} \right) & + \sin \left(2\theta - \frac{4\pi}{3} \right) \\ \sin \left(\theta - \frac{2\pi}{3} \right) & \cos \left(\theta - \frac{2\pi}{3} \right) & \sin \left(2\theta - \frac{4\pi}{3} \right) \end{vmatrix}$$

Now, $\sin \left(\theta + \frac{2\pi}{3} \right) + \sin \left(\theta - \frac{2\pi}{3} \right)$

$$\begin{aligned}
 &= 2 \sin \left(\frac{\theta + \frac{2\pi}{3} + \theta - \frac{2\pi}{3}}{2} \right) \cos \left(\frac{\theta + \frac{2\pi}{3} - \theta + \frac{2\pi}{3}}{2} \right) \\
 &= 2 \sin \theta \cos \frac{2\pi}{3} = 2 \sin \theta \cos \left(\pi - \frac{\pi}{3} \right) = -2 \sin \theta \cos \frac{\pi}{3} = -\sin \theta
 \end{aligned}$$

and

$$\begin{aligned}
 &\cos \left(\theta + \frac{2\pi}{3} \right) + \cos \left(\theta - \frac{2\pi}{3} \right) \\
 &= 2 \cos \left(\frac{\theta + \frac{2\pi}{3} + \theta - \frac{2\pi}{3}}{2} \right) \cos \left(\frac{\theta + \frac{2\pi}{3} - \theta + \frac{2\pi}{3}}{2} \right) \\
 &= 2 \cos \theta \cos \left(\frac{2\pi}{3} \right) = 2 \cos \theta \cos \left(-\frac{1}{2} \right) = -\cos \theta \quad \text{and} \quad \sin \left(2\theta + \frac{4\pi}{3} \right) + \sin \left(2\theta - \frac{4\pi}{3} \right) \\
 &= 2 \sin \left(\frac{2\theta + \frac{4\pi}{3} + 2\theta - \frac{4\pi}{3}}{2} \right) \cos \left(\frac{2\theta + \frac{4\pi}{3} - 2\theta + \frac{4\pi}{3}}{2} \right) \\
 &= 2 \sin 2\theta \cos \frac{4\pi}{3} = 2 \sin 2\theta \cos \left(\pi + \frac{\pi}{3} \right) = -2 \sin 2\theta \cos \frac{\pi}{3} = -\sin 2\theta
 \end{aligned}$$

$$\therefore \Delta = \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ -\sin \theta & -\cos \theta & -\sin 2\theta \\ \sin \left(\theta - \frac{2\pi}{3} \right) & \cos \left(\theta - \frac{2\pi}{3} \right) & \sin \left(2\theta - \frac{4\pi}{3} \right) \end{vmatrix} = 0 \quad [\text{since, } R_1 \text{ and } R_2 \text{ are proportional}]$$

61. Given, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Applying $R_3 \rightarrow R_3 - R_1 - 2R_2$, we get

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} \quad [C_2 \rightarrow C_2 \rightarrow C_1]$$

$$\Rightarrow f'(5/2) = 0 \Rightarrow 5a + b = 0 \quad \dots (i)$$

Also, $f(0) = 2 \Rightarrow c = 2$ and $f(1) = 1$

$$\Rightarrow a + b + c = 1 \quad \dots (ii)$$

On solving Eqs. (i) and (ii) for a, b , we get

$$a = \frac{1}{4}, b = -\frac{5}{4}$$

Thus, $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

62.(0) Since, a, b, c are p th, q th and r th terms of HP.

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in an AP. } \Rightarrow \left. \begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned} \right\} \dots (i)$$

$$\text{Let } \Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{From Eq. (i)}]$$

$$= abc \begin{vmatrix} A + (p-1)D & A + (q-1)D & A + (r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - (A - D)R_3 - DR_2$

$$= abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

63. Given, $a > 0, d > 0$ and let

$$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Taking $\frac{1}{a(a+d)(a+2d)}$ common from R_1 ,

$$\frac{1}{(a+d)(a+2d)(a+3d)} \text{ from } R_2,$$

$$\frac{1}{(a+2d)(a+3d)(a+4d)} \text{ from } R_3$$

$$\Rightarrow \Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

$$\begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \Delta'$$

$$\text{where, } \Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_2$

$$\Rightarrow \Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$\Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)2d & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3 , we get $\Delta' = 2d^2 \begin{vmatrix} a+2d & a \\ d & d \end{vmatrix}$

$$\Delta' = (2d^2)(d)(a+2d-a) = 4d^4$$

$$\therefore \Delta = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

64.

Let $\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$\Rightarrow \Delta = \begin{vmatrix} \cos A \cos P + \sin A \sin P & \cos(A-Q) & \cos(A-R) \\ \cos B \cos P + \sin B \sin P & \cos(B-Q) & \cos(B-R) \\ \cos C \cos P + \sin C \sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} \cos A \cos P & \cos(A-Q) & \cos(A-R) \\ \cos B \cos P & \cos(B-Q) & \cos(B-R) \\ \cos C \cos P & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \begin{vmatrix} \sin A \sin P & \cos(A-Q) & \cos(A-R) \\ \sin B \sin P & \cos(B-Q) & \cos(B-R) \\ \sin C \sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$\Rightarrow \Delta = \cos P \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1 \cos Q, C_3 \rightarrow C_3 - C_1 \cos R$ in first determinant and $C_2 \rightarrow C_2 - C_1 \sin Q$ in second determinant

$$\Rightarrow \Delta = \cos P \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos A \cos Q & \cos A \cos R \\ \sin B & \cos B \cos Q & \cos B \cos R \\ \sin C & \cos C \cos Q & \cos C \cos R \end{vmatrix}$$

$$\Delta = \cos P \sin Q \sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} + \sin P \cos Q \cos R \begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix}$$

$$\Delta = 0 + 0 = 0$$

65. Given, $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$

Taking $n!, (n+1)!$ and $(n+2)!$ common from R_1, R_2 and R_3 , respectively.

$$\therefore D = n!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_2$, we get

$$D = n!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Expanding along C_1 , we get

$$D = (n!)(n+1)!(n+2)![(2n+6) - (2n+4)]$$

$$D = (n!)(n+1)!(n+2)! [2]$$

On dividing both side by $(n!)^3$

$$\Rightarrow \frac{D}{(n!)^3} = \frac{(n!)(n!)(n+1)(n+1)(n+2)2}{(n!)^3}$$

$$\Rightarrow \frac{D}{(n!)^3} = 2(n+1)(n+1)(n+2)$$

$$\Rightarrow \frac{D}{(n!)^3} = 2(n^3 + 4n^2 + 5n + 2) = 2n(n^2 + 4n + 5) + 4$$

$$\Rightarrow \frac{D}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$$

Which shows that $\left[\frac{D}{(n!)^3} - 4 \right]$ is dividing by n .

66. Let $\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}$

Applying $R_1 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\begin{aligned} \Delta &= \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} \\ &= c \begin{vmatrix} a-p & q-b \\ a-p & 0 \end{vmatrix} + (r-c) \begin{vmatrix} p & b \\ a-p & q-b \end{vmatrix} \\ &= -c(a-p)(q-b) + (r-c)[p(q-b) - b(a-p)] \\ &= -c(a-p)(q-b) + p(r-c)(q-b) - b(r-c)(a-p) \end{aligned}$$

Since, $\Delta = 0$

$$\Rightarrow -c(a-p)(q-b) + p(r-c)(q-b) - b(r-c)(a-p) = 0$$

$$\Rightarrow \frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0$$

[on dividing both sides by $(a-p)(q-b)(r-c)$]

$$\Rightarrow \frac{p}{p-a} + \frac{b}{q-b} + 1 + \frac{c}{r-c} + 1 = 2 \Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

67. We know, $A28 = A \times 100 + 2 \times 10 + 8$

$$3B9 = 3 \times 100 + B \times 10 + 9$$

$$\text{and } 62C = 6 \times 100 + 2 \times 10 + C$$

Since, $A28, 3B9$ and $62C$ are divisible by k , therefore there exist positive integers m_1, m_2 and m_3 such that, $100 \times A + 10 \times 2 + 8 = m_1 k, 100 \times 3 + 10 \times B + 9 = m_2 k$ and $100 \times 6 + 10 \times 2 + C = m_3 k \quad \dots (i)$

$$\therefore \Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

Applying $R_2 \rightarrow 100R_1 + 10R_3 + R_2$

$$\Rightarrow \Delta = \begin{vmatrix} A & 3 & 6 \\ 100A + 2 \times 10 + 8 & 100 \times 3 + 10 \times B + 9 & 100 \times 6 + 10 \times 2 + C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix} \quad [\text{from Eq.(i)}]$$

$$= \begin{vmatrix} A & 3 & 6 \\ m_1 k & m_2 k & m_3 k \\ 2 & B & 2 \end{vmatrix} = k \begin{vmatrix} A & 3 & 6 \\ m_1 & m_2 & m_3 \\ 2 & B & 2 \end{vmatrix}$$

$$\therefore \Delta = mk$$

Hence, determinant is divisible by k .

68. Given, $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$\begin{aligned} \therefore \sum_{a=1}^n \Delta_a &= \sum_{a=1}^n \begin{vmatrix} \sum_{a=1}^n (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^n (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} \\ &= \frac{n^2(n-1)}{2} \begin{vmatrix} \frac{(2n-1)}{3} & 2n & 4n-2 \\ \frac{n(n-1)}{2} & 3n^2 & 3n^2-3n \end{vmatrix} \\ &= \frac{n^3(n-1)}{12} \begin{vmatrix} 1 & 1 & 6 \\ 2n-1 & 6n & 12n-6 \\ n-1 & 6n & 6n-6 \end{vmatrix} \end{aligned}$$

Applying $C_3 \rightarrow C_3 - 6C_1$

$$= \frac{n^3(n-1)}{12} \begin{vmatrix} 1 & 1 & 0 \\ 2n-1 & 6n & 0 \\ n-1 & 6n & 0 \end{vmatrix} = 0$$

$$\Rightarrow \sum_{a=1}^n \Delta_a = c \quad [c = 0, \text{ i.e. constant}]$$

69. Let $\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$

Applying $C_3 \rightarrow C_3 + C_2$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix} \quad [\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

Applying $C_2 \rightarrow C_2 + C_1$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

Applying $C_3 \rightarrow C_3 + C_2$

$$\Rightarrow \Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

Hence proved.

70. Since, α is repeated root of $f(x) = 0$.

$$\therefore f(x) = a(x - \alpha)^2, a \in \text{constant } (\neq 0)$$

$$\text{Let } \phi(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

To show $\phi(x)$ is divisible by $(x - \alpha)^2$, it is sufficient to show that $\phi(\alpha)$ and $\phi'(\alpha) = 0$.

$$\therefore \phi(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0 \quad [\because R_1 \text{ and } R_2 \text{ are identical}]$$

$$\text{Assign, } \phi'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$$\phi'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0 \quad [\because R_1 \text{ and } R_3 \text{ are identical}]$$

Thus, α is a repeated root of $\phi(x) = 0$.

Hence, $\phi(x)$ is divisible by $f(x)$.

71. Let $\Delta = \begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$

Applying $R_2 \rightarrow R_2 - (R_1 + R_3)$, we get

$$\Delta = \begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ -4 & 0 & 0 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + \frac{x^2}{4}R_2$ and $R_3 \rightarrow R_3 + \frac{x^2}{4}R_2$, we get

$$\Delta = \begin{vmatrix} x & x + 1 & x - 2 \\ -4 & 0 & 0 \\ 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_1 = \begin{vmatrix} x + 0 & x + 1 & x - 2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

$$= \begin{vmatrix} x & x & x \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} = x \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$\Rightarrow \Delta = Ax + B$$

where, $A = \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$ and $B = \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

72. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a + b + c & b & c \\ a + b + c & c & a \\ a + b + c & a & b \end{vmatrix} = (a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\begin{aligned} &= (a + b + c) \begin{vmatrix} 1 & b & c \\ 0 & c - b & a - c \\ 0 & a - b & b - c \end{vmatrix} \\ &= (a + b + c)[-(c - b)^2 - (a - b)(a - c)] \\ &= -(a + b + c)(a^2 + b^2 + c^2 - ab - ac - ca) \\ &= -\frac{1}{2}(a + b + c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) \\ &= -\frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2] \end{aligned}$$

which is always negative.

73. Since, $AX = U$ has infinitely many solutions,

$$\Rightarrow |A| = 0 \Rightarrow \begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix} = 0$$

$$\Rightarrow a(bc - bd) + 1(d - c) = 0 \Rightarrow (d - c)(ab - 1) = 0$$

$$\therefore ab = 1 \text{ or } d = c$$

$$\text{Again, } |A_3| = \begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix} = 0 \Rightarrow g = h$$

$$\Rightarrow |A_2| = \begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix} = 0 \Rightarrow g = h$$

$$\text{and } |A_1| = \begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix} = 0 \Rightarrow g = h$$

$$\therefore g = h, c = d \text{ and } ab = 1 \quad \dots (i)$$

$$\text{Now, } BX = V$$

$$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0 \quad [\text{from Eq. (i)}]$$

[since, C_2 and C_3 are equal]

$$\therefore BX = V \text{ has no solution.}$$

$$|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0 \quad [\text{from Eq. (i)}]$$

[since, $c = d$ and $g = h$]

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df \quad [\because c = d]$$

$$\text{Since, } adf \neq 0 \Rightarrow |B_2| \neq 0$$

$$|B| = 0 \text{ and } |B_2| \neq 0$$

$$\therefore BX = V \text{ has no solution.}$$

74. Given, $\lambda x + (\sin \alpha)y + (\cos \alpha)z = 0$

$$x + (\cos \alpha)y + (\sin \alpha)z = 0$$

$$\text{and } -x + (\sin \alpha)y - (\cos \alpha)z = 0 \text{ has non-trivial solution.}$$

$$\therefore \Delta = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & \sin \alpha & \cos \alpha \\ 1 & \cos \alpha & \sin \alpha \\ -1 & \sin \alpha & -\cos \alpha \end{vmatrix} = 0$$

$$\Rightarrow \lambda(-\cos^2 \alpha - \sin^2 \alpha) - \sin \alpha(-\cos \alpha + \sin \alpha) + \cos \alpha(\sin \alpha + \cos \alpha) = 0$$

$$\Rightarrow -\lambda + \sin \alpha \cos \alpha + \sin \alpha \cos \alpha - \sin^2 \alpha + \cos^2 \alpha = 0$$

$$\Rightarrow \lambda = \cos 2\alpha + \sin 2\alpha$$

$$[\because -\sqrt{a^2 + b^2} \leq a \sin \theta + b \cos \theta \leq \sqrt{a^2 + b^2}]$$

$$\therefore -\sqrt{2} \leq \lambda \leq \sqrt{2} \quad \dots (i)$$

Again, when $\lambda = 1$, $\cos 2\alpha + \sin 2\alpha = 1$

$$\Rightarrow \frac{1}{\sqrt{2}} \cos 2\alpha + \frac{1}{\sqrt{2}} \sin 2\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos(2\alpha - \pi/4) = \cos \pi/4$$

$$\therefore 2\alpha - \pi/4 = 2n\pi \pm \pi/4$$

$$\Rightarrow 2\alpha = 2n\pi - \pi/4 + \pi/4 \text{ or } 2\alpha = 2n\pi + \pi/4 + \pi/4$$

$$\therefore \alpha = n\pi \text{ or } n\pi + \pi/4$$

75. Since, α_1, α_2 are the roots of $ax^2 + bx + c = 0$.

$$\Rightarrow \alpha_1 + \alpha_2 = -\frac{b}{a} \text{ and } \alpha_1\alpha_2 = \frac{c}{a} \quad \dots (i)$$

Also, β_1, β_2 are the roots of $px^2 + qx + r = 0$.

$$\Rightarrow \beta_1 + \beta_2 = -\frac{q}{p} \text{ and } \beta_1\beta_2 = \frac{r}{p} \quad \dots (ii)$$

Given system of equations

$$\alpha_1 y + \alpha_2 z = 0$$

and $\beta_1 y + \beta_2 z = 0$, has non-trivial solution.

$$\therefore \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0 \Rightarrow \frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}$$

$$\text{Applying componendo-dividendo } \frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} = \frac{\beta_1 + \beta_2}{\beta_1 - \beta_2}$$

$$\Rightarrow (\alpha_1 + \alpha_2)(\beta_1 - \beta_2) = (\alpha_1 - \alpha_2)(\beta_1 + \beta_2)$$

$$\Rightarrow (\alpha_1 + \alpha_2)^2 \{(\beta_1 + \beta_2)^2 - 4\beta_1\beta_2\} = (\beta_1 + \beta_2)^2 \{(\alpha_1 + \alpha_2)^2 - 4\alpha_1\alpha_2\}$$

From Eqs. (i) and (ii), we get

$$\frac{b^2}{a^2} \left(\frac{q^2}{p^2} - \frac{4r}{p} \right) = \frac{q^2}{p^2} \left(\frac{b^2}{a^2} - \frac{4c}{a} \right) \Rightarrow \frac{b^2 q^2}{a^2 p^2} - \frac{4b^2 r}{a^2 p} = \frac{b^2 q^2}{a^2 p^2} - \frac{4q^2 c}{ap^2} \Rightarrow \frac{b^2 r}{a} = \frac{q^2 c}{p} \Rightarrow \frac{b^2}{q^2} = \frac{ac}{pr}$$

76. The system of equations has non-trivial solution, if $\Delta = 0$.

$$\Rightarrow \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Expanding along C_1 , we get $\sin 3\theta \cdot (28 - 21) - \cos 2\theta(-7 - 7) + 2(-3 - 4) = 0$

$$\Rightarrow 7 \sin 3\theta + 14 \cos 2\theta - 14 = 0 \Rightarrow \sin 3\theta + 2 \cos 2\theta - 2 = 0$$

$$\Rightarrow 3 \sin \theta - 4 \sin^3 \theta + 2(1 - 2 \sin^2 \theta) - 2 = 0 \Rightarrow \sin \theta(4 \sin^2 \theta + 4 \sin \theta - 3) = 0$$

$$\Rightarrow \sin \theta(2 \sin \theta - 1)(2 \sin \theta + 3) = 0 \Rightarrow \sin \theta = 0, \sin \theta = \frac{1}{2}$$

[neglecting $\sin \theta = -3/2$]

$$\Rightarrow \theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}, n \in \mathbb{Z}$$

77. The given system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ has atleast one solution, if $\Delta \neq 0$.

$$\therefore \Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} \neq 0$$

$$\Rightarrow 3(2\lambda + 15) + 1(\lambda + 18) + 4(5 - 12) \neq 0 \Rightarrow 7(\lambda + 5) \neq 0 \Rightarrow \lambda \neq -5$$

Let $z = -k$, then equations become $3x - y = 3 - 4k$ and $x + 2y = 3k - 2$

$$\text{On solving, we get } x = \frac{4 - 5k}{7}, y = \frac{13k - 9}{7}, z = k$$

78. Given system of equations are $3x + my = m$ and $2x - 5y = 20$

$$\text{Here, } \Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -15 - 2m \text{ and } \Delta_x = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m, \Delta_y = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix} = 60 - 2m$$

If $\Delta = 0$, then system is inconsistent, i.e. it has no solution.

if $\Delta \neq 0$, i.e. $m \neq \frac{15}{2}$, the system has a unique solution for any fixed value of m .

$$\text{We have, } x = \frac{\Delta_x}{\Delta} = \frac{-25m}{-15 - 2m} = \frac{25m}{15 + 2m}$$

$$\text{and } y = \frac{\Delta_y}{\Delta} = \frac{60 - 2m}{-15 - 2m} = \frac{2m - 60}{15 + 2m}$$

$$\text{For } x > 0, \frac{25m}{15 + 2m} > 0 \Rightarrow m > 0 \text{ or } m < -\frac{15}{2} \quad \dots (i)$$

$$\text{and } y > 0, \frac{2m - 60}{2m + 15} > 0 \Rightarrow m > 30 \text{ or } m < -\frac{15}{2} \quad \dots (ii)$$

$$\text{From Eqs. (i) and (ii), we get } m < -\frac{15}{2} \text{ or } m > 30$$

79. Since, the given system of equations poses non-trivial solution, if $\begin{vmatrix} 0 & 1 & -2 \\ 0 & -3 & 1 \\ k & -5 & 4 \end{vmatrix} = 0 \Rightarrow k = 0$

On solving the equations $x = y = z = \lambda$

$\therefore$ For $k = 0$, the system has infinite solutions of $\lambda \in R$.

80. Given systems of equations can be rewritten as $-x + cy + by = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$

Above system of equations are homogeneous equation.

Since, x, y and z are not all zero, so it has non-trivial solution.

Therefore, the coefficient of determinant must be zero.

$$\therefore \begin{vmatrix} -1 & c & b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow -1(1 - a^2) - c(-c - ab) + b(ca + b) = 0 \Rightarrow a^2 + b^2 + c^2 + 2abc - 1 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$$

81.(BCD) As $\text{adj}(\text{adj}A) = |A|^{n-2} A$

$$\Rightarrow M^{-1} = |M| M \quad \dots\dots\dots(1)$$

$$\Rightarrow \frac{1}{|M|} = |M|^4 \Rightarrow |M|^5 = 1 \Rightarrow |M| = 1 \quad \dots\dots\dots(2)$$

Also pre multiply (1) by M & placing $|M| = 1$

$$M^2 = I \quad \dots\dots\dots(3)$$

Also from (1)

$$\text{Adj}M = M \Rightarrow (\text{Adj}M)^2 = M^2 = I$$

82. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Given : $a + d = 3$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} a^2 + bc & 3b \\ 3c & bc + d^2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^3 = \begin{bmatrix} a^3 + abc + 3bc & a^2b + b^2c + 3bd \\ 3ac + bc^2 + cd^2 & 3bc + bcd + d^3 \end{bmatrix}$$

$$\text{tr}(A^3) = a^3 + abc + 3bc + 3bc + bcd + d^3 = a^3 + bc[a + b + d] + d^3$$

$$\text{tr}(A^3) = a^3 + 9bc + d^3 = -18 = a^3 + 9bc + (3 - a)^3 = -18 = a^3 + 9bc + 27 - a^3 - 27a + 9a^2 = -18$$

$$= 9bc - 27a + 9a^2 = -45 = bc - 3a + a^2 = -5$$

$$\text{Now } |A| = ad - bc = a(3 - a) - bc = 3a - a^2 - bc = 5$$